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Research Paper

Cosine Integral Transform Method for Solving the Westergaard Problem in Elasticity of the Half-Space

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ABSTRACT: The cosine integral transform method is applied to find the expressions for spatial variations of displacements and stresses in the Westergaard continuum under vertical concentrated loading, and distributed loadings acting over lines and geometric areas on the surface. The half-space is considered to be horizontally inextensible and the displacement field reduces to the vertical displacement component. The paper derives a displacement formulation of the equation of equilibrium in the vertical direction. Cosine integral transformation is applied to the formulated equation and the Boundary Value Problem (BVP) is found to simplify to Ordinary Differential Equation (ODE). The general solution of the ODE is obtained in the cosine integral transform space. The requirement of bounded solutions is used to obtain one integration constant. Inversion of the bounded solution gave the solution in the real problem domain space. The stress fields are obtained using the stress-displacement equations. The requirement of equilibrium of the vertical stress fields and the vertical point loading at the origin is used to determine the remaining integration constant, and thus the vertical deflections and the stresses. The solutions obtained are kernel functions employed to derive the expressions for solutions for line, and uniformly distributed loads applied over given geometric areas such as rectangular and circular areas. The vertical stresses are expressed in terms of dimensionless vertical stress influence factors and tabulated. The vertical displacements and stresses obtained are identical with Westergaard solutions obtained by stress function method. The solutions agree with results obtained by Ike using Hankel transform method.

Keywords: Cosine Integral Transform Method, Elastic Half-Space, Inverse Cosine Integral Transform, Stress Fields, Westergaard Problem.

THE ELASTIC HALF-SPACE PROBLEM

This problem is a classical subject in the mathematical theory of elasticity which is concerned with finding spatial variations of stresses and displacements caused by vertical load concentrated at a point, and loads distributed over a known geometric area on the boundary (x_3=0 plane) of the half-space. The elastic half-space occupies the region of space defined mathematically by: $-\infty \leq x_1 \leq \infty$, $-\infty \leq x_2 \leq \infty$, $0 \leq x_3 \leq \infty$. x_1, x_2 and x_3 are the three right handed coordinate

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axes used to describe the three-dimensional (3D) elasticity problem. x_1 and x_2 are the coordinate axes for the horizontal plane with x_1 pointing in the positive direction of the conventional x-Cartesian coordinate axis and x_2 pointing in the positive direction of the conventional y-Cartesian coordinate axis. x_1x_2 coincides with the surface and boundary of the medium. x_3 is the coordinate axis used to describe the vertical axis, and is pointing downwards from the x_1x_2 coordinate plane. x_3 points in the same direction as the z-Cartesian coordinate axis in a right handed xyz-Cartesian coordinate system. Elastic half-space problems are commonly encountered in the advanced theory of materials and solids, advanced mechanics of soils and structures, foundation engineering and geotechnical engineering. Specifically, they arise in problems concerned with the determination of the spatial variations of displacements and stresses and attendant deformations in soil due to boundary foundation loads (Westergaard, 1938; Ike, 2018a,b, 2019a; Fadum, 1948).

The formulation framework guiding such problems are the simultaneous use of the generalized three-dimensional material constitutive relations, the geometrical relations of strains and displacements and the equations of equilibrium in the x_1, x_2, and x_3 Cartesian coordinate axes for a differential element of the half-space. (Ike, 2019a). The resulting equations in 3D problems are fifteen in number, in terms of three unknown displacements, six stress and six strain components. The set of governing equations are usually solved such that the resulting expressions for the unknown spatial variations of stresses and displacements in the 3D medium would satisfy all boundary conditions imposed by applied vertical point and/or distributed boundary loads and also satisfy the conditions of deformations of the half-space.

Solving the system of equations becomes further complicated by considerations of heterogeneity and anisotropy of the half-space material (Ike, 2019a; Liao and Wang, 1998; Tarn and Wang, 1987; Barden, 1963). Even for homogeneous, isotropic cases the system of fifteen equations of elasticity is quite unwieldy to solve in mathematically closed form.

Two basic models of the half-space problems for loads applied on the boundary are the Boussinesq and the Westergaard half-space problems. The Boussinesq half-space problem involves the determination of the spatial variations of stresses and displacements caused by vertical concentrated load at a point on the boundary. The Boussinesq half-space is assumed to have material elastic properties that do not exhibit spatial and/or directional variation at any point in the half-space. The assumption that the material elastic properties of the Boussinesq half-space do not have directional variation at any point is the property of isotropy. The assumption that the material properties do not have spatial variation from one point to another in the half-space is called homogeneity. The Boussinesq half-space is also assumed to be semi-infinite in extent; and defined mathematically using inequalities as: $-\infty \leq x_1 \leq \infty$; $-\infty \leq x_2 \leq \infty$; $0 \leq x_3 \leq \infty$. The problem is a significant theme in the classical theory of elasticity of three-dimensional (3D) media because the solutions are fundamental or Kernel functions (or Green functions) which are very important in the derivation of mathematical expressions for the spatial variations of the normal, shear stresses and displacements in the half-space medium caused by loads that have known distributions over geometric areas on the surface. The geometric areas can be one-dimensional lines of finite or infinite length or two-dimensional surfaces which can be rectangular, circular, elliptical in shape or combinations of such regular geometric shapes.

In the Westargaard problem, the stress and displacement fields are to be determined in an elastic half-space that is considered to be internally reinforced by several layers of negligibly small thickness but are infinitely rigid. This thus prevents the medium from undergoing displacement in all the horizontal directions where the x_3-Cartesian coordinate direction points downwards from the soil surface (VSUT, 2018).

The basic assumptions in the Westergaard half-space formulation are as follows: (Ike, 2018a,b, 2019a, Rocscience, 2018):

i) The infinitely rigid strata are so closely spread that the aggregate elastic and physical properties are approximately assumed to exhibit no directional and spatial variation.

ii) The thicknesses of the infinitely rigid layers are negligibly small when compared to the thicknesses of the alternating weaker layers.

iii) The infinitely rigid layers are horizontally not capable of extension and horizontally rigid in the x_1 and x_2 coordinate axes. Consequent to the non-extensibility and rigidity, in the x_1 and x_2 coordinates, the x_1 and x_2-Cartesian components of the displacement field would both vanish. Mathematically, $u_1=u_2=0$ in which u_1 is the x_1 – displacement field component and u_2 is the x_2 – displacement field component.

iv) The horizontally non-extensible (or horizontally rigid) infinitely rigid layers prevent the development of normal strains in the x_1 and x_2 coordinate axes in the soft and infinitely rigid layers. Consequently, strain components in both the x_1 and x_2 coordinate axes would vanish. Thus $\varepsilon_{11}=\varepsilon_{22}=0$ where ε_{11} is the normal strain in the x_1 direction and ε_{22} is the normal strain in the x_2 direction. The shear strain field γ_{12} at any point in the half-space would consequently vanish, and this conclusion follows from the shear strain-displacement equation for infinitesimal displacement elasticity theory (Anyaegbunam et al., 2011; Ike, 2019a).

Soils that are classified as sedimentary soils of which natural clay is an example are generally anisotropic in nature (Bowles, 1997; Ike, 2019a). The Westergaard half-space model appears in some instances to be a better model of the soil mass represented as a semi-infinite medium (Rocscience, 2018).

Simplified formulations of the elastic half-space problems have been presented using three basic methods namely – displacement, stress and mixed (hybrid) methods (Sadd, 2014; Bowles, 1997; Barbar, 2010; Ojedokun and Olutego, 2012; Ike, 2019a; Chau, 2013).

The displacement method which is used to formulate the Westergaard problem of elasticity in this work is originally due to Navier and Lamé. The displacement formulation of the problem entails expressing the set of fifteen governing equations of the half-space using three Cartesian components of the displacement field as the primary unknowns (Kachanov et al., 2003; Sitharam and GovindaRaju, 2017; Palaniappan, 2011; Hazel, 2015; Ike, 2019a).

One merit of the Navier – Lamé displacement technique is that the governing equations are reduced from fifteen in a 3D problem to three. The remaining unknowns in a displacement formulation which are the Cauchy stresses and strains are obtained from the displacements (once solved) using the kinematic relations, and the material constitutive equations. The main advantage of the displacement method in solving elastic half-space problems is thus the reduction in the analytical rigour offered by the considerable decrease in the governing equations to be solved for 3D problems; in both numbers and complexities.

Stress based methods, originally due to Michell and Beltrami, are based on reformulating the set of fifteen governing equations of the elastic half-space using the six Cauchy stress field components for a three dimensional problem (Ike, 2019a, 2018a,b,c; Ojedokun and Olutoge, 2012; Barbar, 2010;

Green and Zerna, 1954). The merit is the decrease in the equations for solution from fifteen to six for a 3D problem. The system of coupled Beltrami-Michell partial differential equations expressed using normal and shear stresses are integrated to obtain stress field components. The normal and shear strains are then obtained from the generalized Hooke's law relating stresses and strains. The displacements are determined using the kinematic relations. The stress functions are scalar potential functions of the space coordinates that satisfy the stress formulation of the differential equations of equilibrium (Ike, 2018c). The stress fields in the elasticity problem could be derived in terms of spatial derivatives of the stress function.

Advantages of the scalar stress function method include:

i) The stress function is usually a biharmonic function and could readily be obtained from a catalogue of biharmonic functions in advanced mathematics.

ii) The stress function could be obtained as a polynomial that is biharmomic.

iii) Stress fields satisfy the requirements of equilibrium and compatibility of strain fields.

v) Additionally, the scalar stress function method has the merit that scalars are much easier to handle than vectors, and are easily amenable to coordinate transformations.

The mixed method of formulation, though rarely applied for finding solutions of elastic half-space problems, involve reformulating the governing equations in such a way that the equations are expressed using some stresses and displacements as unknowns to be determined for solution (Ike, 2019a). Analytical expressions that satisfy the Navier-Lamé displacement equations of static equilibrium at all points in an elastic half-space have been developed and presented respectively as displacement functions and stress functions. The mixed method can be considered a hybrid of the stress-based and the displacement-based methods (Nwoji et al., 2017a; Nwoji et al., 2017b; Ike, 2017; Ike et al., 2017a,b; Apostol, 2017; Lurie and Vasilev, 1995; Chau, 2013; Teodorescu, 2013; Abeyaratne, 2012; Davis and Selvadurai, 1996; Podio-Guidugli and Favata, 2014). Displacement potential functions found to have been previously used by researchers include: Boussinesq, Papkovich-Neuber, Trefftz, Green and Zerna, and Cerrutti functions.

Some of the stress functions used in the stress-based methods were derived by: Airy, Morera, Maxwell, Love, Boussinesq, Papkovich-Neuber, and are respectively named after them.

Miroshnikov (2018) presented an analytical and numerical solution to the theory of elasticity problem where stresses are known on a boundary for a homogeneous half-space with several cylindrical cavities parallel to each other and the boundary of the half-space. The specified boundary stresses are assumed to rapidly decay to zero at great distances from the origin of coordinates on the boundaries of the cavities and on the boundary of the half-space. Miroshnikov (2018) used the generalized Fourier method in relation to the system of Lamé equations in the cylindrical coordinates system. The resulting infinite number of system of linear algebraic equations was solved by finite truncation to obtain the displacements and stresses in the elastic body.

Zhou and Gao (2013) used the Papkovich-Neuber harmonic functions, Fourier transformation technique and the cylindrical functions for formulating elastic half-space and half-plane problems. They then derived mathematical solutions for such problems for the case of vertically applied load with known distribution over a geometric area. They obtained solutions that reduced to the solution for half-space and half-plane contact problems of classical linear elasticity if the surface effects are disregarded.

Dobrovolsky (2015) considered the

inhomogeneous isotropic elastic half-space problem with one-dimensional continuous heterogeneity; and obtained solutions using two-dimensional Fourier transformation.

Hayati et al. (2013) used new scalar harmonic functions to derive the dynamic Green's functions for the axially symmetric problem formulated using Biot's theory. The harmonic potential function used uncoupled the equation of the thermoelastic half-space resulting to a sixth-order PDE expressed using cylindrical polar coordinates. They applied Fourier–Bessel transformation with respect to the radial coordinate to obtain an ODE of the sixth order. They proceeded to solve the ODE to satisfy the boundary conditions thus obtaining displacements, spatial variations of stresses and temperature fields in the Fourier–Bessel transformed space. They used inverse Fourier–Bessel transformation for obtaining solutions for the real actual problem which were validated by showing agreement with previously obtained solutions in the literature.

Other research papers that recorded significant contributions to the theme of elastic half-space theory for isotropic, orthotropic, transversely isotropic, homogeneous, inhomogeneous elastostatics and elasto-dynamic cases include: Naeeni and Eskandari-Ghadi (2016); Eskandari-Ghadi et al. (2014); Ardeshir-Behrestaghi et al. (2013); Godara et al. (2017) and Miroshnikov (2017).

Integral transform methods that have been used to solve elasticity problems in 2D and 3D include: Hankel (Fourier–Bessel) transform, Fourier transform, Mellin transform and Elzaki transform methods.

Hankel transformation methodology was used for obtaining solutions for the spatial distribution of normal and shear stresses and displacement fields for the Westergaard problem for various boundary loads by Ike (2019a). He used the Hankel transform method, which is an integral transform technique that has the Bessel function as the integral kernel for obtaining general solutions to the spatial variations of displacement and stresses in a Westergaard half-space carrying vertically applied concentrated load on the boundary surface. He then used the resulting solution as fundamental kernel functions to obtain solutions for the spatial distribution of vertical stresses caused by uniformly distributed line loads of finite length applied on the boundary and acting parallel to the *y* axis.

Ike (2019a) similarly used the fundamental solutions of the vertical point load to obtain vertical stresses due to uniformly distributed loads applied over given rectangular areas on the boundary of the Westergaard half-space. He calculated and presented tabulations of influence values for easy calculation of spatial distributions of vertical stresses caused by vertical concentrated loads applied at a reference point on the boundary surface of the half-space. The Hankel transform methodology is especially suitable for boundary value problems formulated in cylindrical polar coordinates. The Hankel (Fourier–Bessel) transform method is also suitable for BVP of mathematics that are symmetric about the *z*-coordinate axis for problems described using $r\theta z$ coordinates (Piessons, 2000; Yokoyama, 2014; Tuteja et al., 2014; Voegtle, 2017; Andrews and Shivamoggi, 1999; Naeeni et al., 2015).

Sneddon (1992, 2010) presented the mathematics of Fourier transformation methods and explained how the method could be applied to the solution of problems in mathematical physics, including some problems in elasticity.

Ike (2020a) used the Elzaki transformation methodology for obtaining the mathematical expressions for the spatial variations of normal and shear stresses and displacements caused by loads applied to the boundary surface. He considered two-dimensional elasticity problems formulated in the plane polar coordinates system. He solved the Flamant problem and problem of strip loads of infinite extent on the half-plane and obtained expressions for stresses that were identical to those in the literature, thus validating the work.

In a related work, Ike (2020b) used the Fourier Cosine integral transform method for obtaining the analytical expressions for the spatial distributions of normal and shear stresses and displacements caused by a vertical concentrated force applied to a reference point on the bounding surface of an elastic half-plane. The half-plane was considered isotropic and homogeneous and the problem assumed to be two-dimensional.

Ike (2019b) used the Mellin transform method to solve the two-dimensional elasticity problem expressed in plane polar coordinates. He considered a vertically applied force at a reference point on an isotropic, homogeneous semi-infinite plane. He obtained solutions for the Flamant problem that were identical with previous solutions in the literature, thus validating the study.

However, to the best of the author's knowledge there is no previous presentation of any application of the cosine transformation method in the rigorous way done here for the analysis of the Westergaard problem, which is the subject matter of this research.

In this work, the Westergaard problem in the theory of elasticity of horizontally inextensible elastic half-space is presented using displacement formulation method.

The formulated partial differential equation is solved by cosine integral transformation method.

Purpose and Objectives of Study

The fundamental purpose is to present the application of the cosine integral transform method for solving the Westergaard problem of vertical concentrated force of known magnitude at a reference point on the boundary of a Westergaard half-space that is assumed to be homogeneous, isotropic and linear elastic $(-\infty \leq x_1 \leq \infty;\ -\infty \leq x_2 \leq \infty;\ 0 \leq x_3 \leq \infty)$. The specific objectives are as follows:

i) To derive the equation for static equilibrium using the Cauchy-Navier displacement formulation that accounts for the assumptions of the Westergaard half-space model. The objective is to obtain the Cauchy-Navier displacement equation of equilibrium in the vertical direction for a vertical concentrated force of magnitude denoted by P_0 applied at a reference point O, on the horizontal boundary surface (x_1x_2 plane) of the half-space. The half-space is occupying the three dimensional region R^3 defined mathematically by $-\infty \leq x_1 \leq \infty;$ $-\infty \leq x_2 \leq \infty;\ 0 \leq x_3 \leq \infty.$

ii) To apply the cosine integral transformation to the governing equation, and thus obtain by solving the resulting integral equation, bounded general solutions in the transformed space for the spatial distributions of vertical displacement caused by the vertical concentrated force acting at a reference point on the boundary surface (x_1x_2 plane).

iii) To apply the cosine integral transformed vertical stress boundary condition for determining the integration constant in the bounded general solution for the spatial distributions of vertical displacement caused by vertical concentrated force acting at a reference point on the boundary surface (x_1x_2 plane).

iv) To obtain spatial distributions of normal and shear stresses for the Westergaard problem by use of the kinematic relations; and then determine the vertical stress fields in terms of dimensionless factors.

v) To use the expressions for spatial distribution of stresses and displacements due to a vertical concentrated force applied at a reference point on the surface (x_1x_2 plane) of the Westergaard half-space as Green (fundamental or Kernel) functions; and find by employment of the superposition theory for linear elastic bodies, the expressions for spatial distribution of stresses and displacements in the Westergaard half-space resulting from uniformly distributed loads over lines and geometric areas on the x_1x_2 plane.

vi) To use the expressions for the spatial distribution of stresses caused by a vertical concentrated force applied at a reference point on the boundary surface (x_1x_2 plane) as Green (fundamental or Kernel) function and determine by employment of the superposition theory for linear elasticity expression for the variation of vertical stresses with depth under a corner of a rectangular foundation area of plan dimensions $L\times B$ subjected to uniform distribution of load of known intensity, p_0.

vii) To use the expressions for the spatial distribution of vertical stress caused by a vertical concentrated force acting at a reference point on the boundary (x_1x_2 plane) of the Westergaard half-space as a fundamental function for obtaining using superposition principles the vertical stress field for points under the center of circular foundation carrying uniform load.

THEORETICAL FRAMEWORK

Displacement Formulation of the Westergaard Problem

A displacement formulation is presented for the Westergaard problem by considering the foundational equations of 3D elasticity theory.

Equations of Three-Dimensional (3D) Elasticity Theory

The equations of 3D elasticity theory for static loads are the differential equations of static equilibrium, the generalised Hooke's laws, and the kinematic relations. The three basic set of equations are solved such that all boundary conditions are satisfied.

Differential Equations of Static Equilibrium

The differential equations of static equilibrium for elastostatic problems are given by (Ike et al., 2017b; Nwoji et al., 2017a,b):

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + f_1 = 0 \quad (1)$$

$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + f_2 = 0 \quad (2)$$

$$\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + f_3 = 0 \quad (3)$$

in which σ_{11}, σ_{22}, σ_{33}: are normal stresses, σ_{12}, σ_{21}, σ_{23}, σ_{32}, σ_{31}, σ_{13}: are shear stresses. f_1, f_2 and f_3: are body force components in the x_1, x_2 and x_3 coordinate directions.

Generalised Hooke's Laws

The generalised Hooke's laws for homogeneous, isotropic, linear elastic problems are given by the set of six equations (Ike, 2018a; Nwoji et al., 2017a,b):

$$\sigma_{11} = 2G\varepsilon_{11} + \lambda\varepsilon_v \quad (4)$$

$$\sigma_{22} = 2G\varepsilon_{22} + \lambda\varepsilon_v \quad (5)$$

$$\sigma_{33} = 2G\varepsilon_{33} + \lambda\varepsilon_v \quad (6)$$

$$\sigma_{13} = \sigma_{31} = G\gamma_{13} \quad (7)$$

$$\sigma_{12} = \sigma_{21} = G\gamma_{12} \quad (8)$$

$$\sigma_{23} = \sigma_{32} = G\gamma_{23} \quad (9)$$

where $$\varepsilon_v = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \quad (10)$$

where ε_v: is the volumetric strain, ε_{11}, ε_{22}, ε_{33}: are normal strains, γ_{12}, γ_{13}, γ_{23}: are shear strains, G: is the shear modulus and λ: is the Lamé coefficient or modulus.

The shear modulus, also called the rigidity modulus, G, is expressed in terms of the

Young's modulus, E and the Poisson's ratio μ by Eq. (11):

$$G = \frac{E}{2(1+\mu)} \tag{11}$$

Kinematic Equations for Infinitesimal Displacement Elasticity

For small displacement assumptions, the strain-displacement relations are given by (Ike et al., 2017b; Nwoji et al., 2017a,b; Ike, 2018b):

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \tag{12}$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \tag{13}$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} \tag{14}$$

$$\gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \tag{15}$$

$$\gamma_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \tag{16}$$

$$\gamma_{13} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \tag{17}$$

in which u_1, u_2 and u_3: are the components of the displacement field in the x_1, x_2 and x_3 Cartesian coordinate directions, respectively.

Governing Equations of the Westergaard Problem

The Westergaard problem shown in Figure 1 involves finding the normal and shear stresses and displacement field component in an elastic half-space with horizontal inextensibility due to boundary vertical concentrated force acting vertically downward at a reference point on the x_1x_2 plane of the half-space $-\infty \le x_1 \le \infty$, $-\infty \le x_2 \le \infty$, $0 \le x_3 \le \infty$.

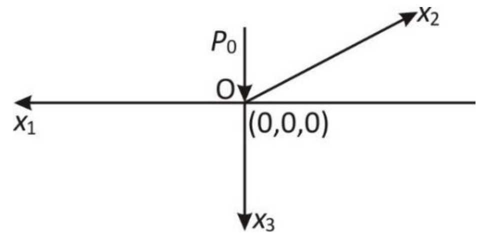

Fig. 1. Vertical concentrated load P_0 at the origin of a Westergaard half-space

In the Westergaard problem there are no displacement components in the x_1 and x_2 coordinate axes. The non-zero displacement component is the vertical displacement component (x_3 coordinate component of displacement).

The displacement field components for the Westergaard problem are:

$$u_1 = 0,\ u_2 = 0 \tag{18}$$

$$u_3(x_1, x_2, x_3) \neq 0 \tag{19}$$

Using Eq. (18) in the strain – displacement relations the strains are obtained as follows:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0 \tag{20}$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = 0 \tag{21}$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} \tag{22}$$

$$\gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0 \tag{23}$$

$$\gamma_{13} = \frac{\partial u_3}{\partial x_1} \tag{24}$$

$$\gamma_{23} = \frac{\partial u_3}{\partial x_2} \tag{25}$$

The stresses are obtained in terms of displacements by using Eqs. (20-25) in the stress-strain relations. The stress-displacement equations are:

$$\sigma_{11} = \lambda \frac{\partial u_3}{\partial x_3} \tag{26}$$

$$\sigma_{22} = \lambda \frac{\partial u_3}{\partial x_3} \quad (27)$$

$$\sigma_{33} = (\lambda + 2G) \frac{\partial u_3}{\partial x_3} \quad (28)$$

$$\sigma_{12} = 0 \quad (29)$$

$$\sigma_{13} = G \frac{\partial u_3}{\partial x_1} \quad (30)$$

$$\sigma_{23} = G \frac{\partial u_3}{\partial x_2} \quad (31)$$

The Cauchy-Navier displacement formulation for equilibrium in the vertical direction is given by substitution of Eqs. (26-31) in the differential equation of equilibrium in the x_3 coordinate direction when body forces are absent or neglected to obtain Eq. (32):

$$G\left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2}\right) + (\lambda + 2G)\frac{\partial^2 u_3}{\partial x_3^2} = 0 \quad (32)$$

or

$$\left(\frac{G}{\lambda + 2G}\right)\left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2}\right) + \frac{\partial^2 u_3}{\partial x_3^2} = 0 \quad (33)$$

$$\text{or } \beta^2 \left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2}\right) + \frac{\partial^2 u_3}{\partial x_3^2} = 0 \quad (34)$$

$$\text{where } \beta^2 = \frac{G}{\lambda + 2G} = \frac{1 - 2\mu}{2(1 - \mu)} \quad (35)$$

Hence Eq. (32) is given in compact form as Eq. (36):

$$\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + \frac{1}{\beta^2}\frac{\partial^2 u_3}{\partial x_3^2} = 0 \quad (36)$$

METHODOLOGY

Cosine Integral Transformation of the Governing Domain Equation

The cosine integral transform is taken of the governing domain equation, Eq. (36) to obtain the integral equation in Eq. (37):

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + \frac{1}{\beta^2}\frac{\partial^2 u_3}{\partial x_3^2}\right) \times \cos(\alpha_1 x_1)\cos(\alpha_2 x_2)\, dx_1 dx_2 = 0 \quad (37)$$

where α_1 and α_2: are the cosine integral transform parameters.

By the linearity property of the cosine integral transformation and the application of the Leibnitz rule, we obtain after simplification, of Eq. (37), the following equation:

$$\frac{1}{\beta^2}\frac{d^2}{dx_3^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_3(x_1, x_2, x_3)\cos(\alpha_1 x_1)\cos(\alpha_2 x_2)\, dx_1 dx_2 - (\alpha_1^2 + \alpha_2^2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_3(x_1, x_2, x_3)\cos(\alpha_1 x_1)\cos(\alpha_2 x_2)\, dx_1 dx_2 = 0 \quad (38)$$

Let

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_3(x_1, x_2, x_3)\cos(\alpha_1 x_1)\cos(\alpha_2 x_2) dx_1 dx_2 = \bar{U}_3(\alpha_1, \alpha_2, x_3) \quad (39)$$

$\bar{U}_3(\alpha_1, \alpha_2, x_3)$: is the cosine integral transform of the vertical displacement $u_3(x_1, x_2, x_3)$.

Then, the second order ordinary differential equation (ODE) is obtained by Eq. (40).

$$\frac{1}{\beta^2}\frac{d^2}{dx_3^2}\bar{U}_3(\alpha_1, \alpha_2, x_3) - (\alpha_1^2 + \alpha_2^2)\bar{U}_3(\alpha_1, \alpha_2, x_3) = 0 \quad (40)$$

or

$$\frac{d^2}{dx_3^2}\bar{U}_3(\alpha_1, \alpha_2, x_3) - \beta^2(\alpha_1^2 + \alpha_2^2)\bar{U}_3(\alpha_1, \alpha_2, x_3) = 0 \quad (41)$$

Using the method of trial functions, or differential operator techniques, the general solution to Eq. (41) is obtained as Eq. (42):

$$\bar{U}_3(\alpha_1,\alpha_2,x_3) = c_1\exp\left(-\beta\sqrt{(\alpha_1^2+\alpha_2^2)}x_3\right) + c_2\exp\left(\beta\sqrt{(\alpha_1^2+\alpha_2^2)}x_3\right) \tag{42}$$

where c_1 and c_2: are the two constants of integration.

For finite values of the vertical displacement $u_3(x_1,x_2,x_3)$ as $x_3\to\infty$, $\bar{U}_3(\alpha_1,\alpha_2,x_3)$ is expected to be finite and bounded as $x_3\to\infty$.

For finite, bounded solutions,

$$c_2 = 0 \tag{43}$$

The bounded solution in the cosine integral transform space is:

$$\bar{U}_3(\alpha_1,\alpha_2,x_3) = c_1\exp\left(-\beta\sqrt{(\alpha_1^2+\alpha_2^2)}x_3\right) \tag{44}$$

By inversion, the bounded solution for the vertical displacement is given by Eq. (45):

$$u_3(x_1,x_2,x_3) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{U}_3(\alpha_1,\alpha_2,x_3)\times \cos(\alpha_1x_1)\cos(\alpha_2x_2)\,d\alpha_1d\alpha_2 \tag{45}$$

Hence, substitution of Eq. (44) in Eq. (45) gives Eq. (46):

$$u_3(x_1,x_2,x_3) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}c_1\exp\left(-\beta\sqrt{(\alpha_1^2+\alpha_2^2)}x_3\right)\times \cos(\alpha_1x_1)\cos(\alpha_2x_2)\,d\alpha_1d\alpha_2 \tag{46}$$

The double cosine integral transform inverse given in Eq. (46) involves a very complicated integration problem. The integration problem in Eq. (46) could be solved by expressing it as a single Hankel transform inversion integral by the aid of coordinate transformation (Sneddon, 2010; 1992) and calculating the resulting Hankel inversion integral. However, the solution of the transform inverse adopted in this work is based on Eq. (36) which is a Laplacian in terms of x_1, x_2, and βx_3.

From, Eq. (36) we simplify to express as the following Laplacian in Eq. (47):

$$\frac{\partial^2u_3}{\partial x_1^2}+\frac{\partial^2u_3}{\partial x_2^2}+\frac{1}{\beta^2}\frac{\partial^2u_3}{\partial x_3^2} = \frac{\partial^2u_3}{\partial x_1^2}+\frac{\partial^2u_3}{\partial x_2^2}+\frac{\partial^2u_3}{\partial(\beta x_3)^2} = \nabla^2u_3(x_1,x_2,\beta x_3) = 0 \tag{47}$$

The general solution to Eq. (47) which should coincide with the solution for Eq. (46) is given by Eq. (48) where $\hat{c}_1$: is a constant that is related to c_1. The relationship between $\hat{c}_1$ and c_1 could be the subject of further research for mathematics scholars.

$$u_3(x_1,x_2,x_3) = \hat{c}_1\frac{1}{R} = \hat{c}_1\frac{1}{(x_1^2+x_2^2+\beta^2x_3^2)^{1/2}} \tag{48}$$

RESULTS

Solution for Vertical Concentrated Load P_0 Applied At a Reference Point on the Westergaard Half-Space

For the case of a vertical point load P_0 applied at a reference point on the bounding surface of the Westergaard half-space as illustrated in Figure 1, the vertical stress field is obtained by substitution of Eq. (48) in Eq. (28) to obtain Eq. (49) which follows:

$$\sigma_{33}(x_1,x_2,x_3) = (\lambda+2G)\frac{\partial u_3}{\partial x_3} = (\lambda+2G)\frac{\partial}{\partial x_3}\hat{c}_1(x_1^2+x_2^2+\beta^2x_3^2)^{-1/2} \tag{49}$$

Thus, performing the partial differentiation problem in Eq. (49) and simplifying gives the following Eq. (50):

$$\sigma_{33}(x_1, x_2, x_3) = -(\lambda + 2G)\hat{c}_1 x_3 \beta^2 \times (x_1^2 + x_2^2 + \beta^2 x_3^2)^{-3/2} \quad (50)$$

The vertical stress field is obtained in terms of an unknown constant of integration $\hat{c}_1$, which is determined by enforcing boundary conditions. The boundary conditions used to determine $\hat{c}_1$ is the requirement of equilibrium of resultant vertical stress fields and the applied vertical point loading.

Thus, we have the equation of equilibrium in the x_3 direction as the following Eq. (51):

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma_{33}(x_1, x_2, x_3) dx_1 dx_2 + P_0 = 0 \quad (51)$$

Substituting the expression for $\sigma_{33}(x_1, x_2, x_3)$ into Eq. (51) we obtain the following Eq. (52):

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -(\lambda + 2G)\hat{c}_1 \beta^2 x_3 (x_1^2 + x_2^2 + \beta^2 x_3^2)^{-3/2} dx_1 dx_2 + P_0 = 0 \quad (52)$$

Re-arranging Eq. (52) we have Eq. (53) as follows:

$$P_0 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\lambda + 2G)\hat{c}_1 \beta^2 x_3 (x_1^2 + x_2^2 + \beta^2 x_3^2)^{-3/2} dx_1 dx_2 \quad (53)$$

Factoring out the constants, a simplification of Eq. (53) is obtained as follows:

$$P_0 = (\lambda + 2G)\hat{c}_1 \beta^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_3 (x_1^2 + x_2^2 + \beta^2 x_3^2)^{-3/2} dx_1 dx_2 \quad (54)$$

The integration problem presented in Eq. (54) is evaluated by the method of change of variables from the Cartesian to the cylindrical polar coordinates. The transformation equations are given by Eqs. (55-57) as follows:

$$x_1 = r\cos\theta \quad (55)$$

$$x_2 = r\sin\theta \quad (56)$$

$$x_3 = z \quad (57)$$

and $0 \le r \le \infty$, $0 \le z \le \infty$, $0 \le \theta \le 2\pi$

then, $dx_1 dx_2 = |J| dr d\theta$ (58)

$$\text{where, } |J| = \left|\frac{\partial(x_1, x_2)}{\partial(r, \theta)}\right| = \begin{vmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \theta} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \theta} \end{vmatrix} \quad (59a)$$

$$|J| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \quad (59b)$$

$|J|$: is the Jacobian of the coordinate transformation from 3D Cartesian to 3D polar coordinates system.

Then, the integration problem in Eq. (54) is expressed using the cylindrical polar coordinates system (Eq. (60)) as follows:

$$P_0 = (\lambda + 2G)\hat{c}_1 \beta^2 \int_0^{2\pi}\int_0^{\infty} z(r^2 + \beta^2 z^2)^{-3/2} r dr d\theta \quad (60)$$

Simplifying, Eq. (60) by performing the integration with respect to θ, the simple Eq. (61) is obtained as follows:

$$P_0 = (\lambda + 2G)\hat{c}_1 \beta^2 \cdot 2\pi \int_0^{\infty} z(r^2 + \beta^2 z^2)^{-3/2} r dr \quad (61)$$

The integral in Eq. (61) is evaluated again by the change of variables from r to $a_0(r)$ where

$$a_0(r) = r^2 + \beta^2 z^2 \tag{62}$$

then, $rdr = \dfrac{da_0}{2}$ (63)

and,

$$P_0 = 2\pi(\lambda + 2G)\hat{c}_1\beta^2 z \int_{r=0}^{r=\infty} (a_0(r))^{-3/2} \frac{da_0(r)}{2} \tag{64}$$

By integration of Eq. (64), Eq. (65) is obtained:

$$P_0 = \frac{2\pi(\lambda + 2G)\hat{c}_1\beta^2 z}{2}\left[\frac{(a_0(r))^{-1/2}}{-1/2}\right]_{r=0}^{r=\infty} \tag{65}$$

By simplifying Eq. (65), Eq. (66) is obtained:

$$P_0 = -2\pi(\lambda + 2G)\hat{c}_1\beta^2 z\left[(r^2 + \beta^2 z^2)^{-1/2}\right]_0^{\infty} \tag{66}$$

By solving Eq. (66), the unknown integration constant is obtained as given by Eq. (67):

$$\hat{c}_1 = \frac{P_0}{2\pi(\lambda + 2G)\beta} \tag{67}$$

Hence, the vertical displacement is obtained as expressed by Eq. (68):

$$u_3(x_1, x_2, x_3) = \frac{P_0}{2\pi\beta(\lambda + 2G)} (x_1^2 + x_2^2 + \beta^2 x_3^2)^{-1/2} \tag{68}$$

Alternatively, Eq. (68) could be expressed as Eqs. (69a), (69b) or (69c):

$$u(x_1, x_2, x_3) = \frac{P_0\beta}{2\pi GR} = \frac{P_0\beta}{2\pi G}(x_1^2 + x_2^2 + \beta^2 x_3^2)^{-1/2} \tag{69a}$$

$$u_3(x_1, x_2, x_3) = \frac{P_0\beta 2(1+\mu)}{2\pi ER} = \frac{P_0(1+\mu)\beta}{E\pi R} \tag{69b}$$

Thus,

$$u_3(x_1, x_3, x_3) = \frac{P_0\beta(1+\mu)}{E\pi}(x_1^2 + x_2^2 + \beta^2 x_3^2)^{-1/2} \tag{69c}$$

Stress Fields

The vertical stress field in the Westergaard half-space is obtained as:

$$\sigma_{33}(x_1, x_2, x_3) = (\lambda + 2G)\frac{\partial u_3}{\partial x_3} = G\beta^{-2}\frac{\partial u_3}{\partial x_3} \tag{70}$$

Substitution of Eq. (69) for $u_3(x_1, x_2, x_3)$ gives:

$$\sigma_{33}(x_1, x_2, x_3) = \frac{P_0(x_1^2 + x_2^2 + \beta^2 x_3^2)^{-1/2}}{2\pi\beta(\lambda + 2G)} \tag{71}$$

Hence,

$$\sigma_{33}(x_1, x_2, x_3) = \frac{P_0}{2\pi\beta}\frac{\partial}{\partial x_3}(x_1^2 + x_2^2 + \beta^2 x_3^2)^{-1/2} \tag{72}$$

Evaluation gives:

$$\sigma_{33}(x_1, x_2, x_3) = \frac{-P_0\beta x_3}{2\pi}(x_1^2 + x_2^2 + \beta^2 x_3^2)^{-3/2} \tag{73}$$

Hence,

$$\sigma_{33}(x_1, x_2, x_3) = \frac{-P_0\beta x_3}{2\pi R^3} = \frac{-P_0}{z^2}\frac{\beta}{2\pi}\left(\beta^2 + \frac{r^2}{z^2}\right)^{-3/2} \tag{74}$$

Alternatively,

$$\sigma_{zz} = \frac{-P_0}{z^2} I_w(r, z, \beta(\mu)) \tag{75}$$

where $I_w(r, z, \beta)$: is the Westergaard vertical stress influence factor for vertical concentrated load acting at the origin of a Westergaard half-space.

The normal stresses in the x_1 and x_2 Cartesian directions are obtained as:

$$\sigma_{11} = \sigma_{22} = \frac{\mu\sigma_{33}}{1-\mu} = k_0\sigma_{33} \tag{76}$$

where k_0: is the coefficient of lateral stress at rest, $k_0 = \mu/(1-\mu)$.

$$\text{or, } \sigma_{11} = \sigma_{22} = \frac{-\mu}{1-\mu}\frac{P_0\beta x_3}{2\pi R^3} \tag{77}$$

The shear stresses σ_{32} are obtained as Eq. (29):

$$\sigma_{32} = G\frac{\partial u_3}{\partial x_2} = G\frac{\partial}{\partial x_2}\frac{P_0\beta}{2\pi G R} \tag{78}$$

$$\sigma_{32} = \frac{GP_0\beta}{2\pi G}\frac{\partial}{\partial x_2}(x_1^2 + x_2^2 + \beta^2 x_3^2)^{-1/2} \tag{79}$$

Hence,

$$\sigma_{32} = \frac{-P_0\beta x_2}{2\pi}(x_1^2+x_2^2+\beta^2 x_3^2)^{-3/2} = \frac{-P_0\beta x_2}{2\pi R^3} \tag{80}$$

$$\sigma_{31} = G\frac{\partial u_3}{\partial x_1} = G\frac{\partial}{\partial x_1}\frac{P_0\beta}{2\pi G R} \tag{81}$$

$$\sigma_{31} = \frac{P_0\beta}{2\pi}\frac{\partial}{\partial x_1}(x_1^2 + x_2^2 + \beta^2 x_3^2)^{-1/2} \tag{82}$$

$$\sigma_{31} = \frac{-P_0\beta x_1}{2\pi}(x_1^2+x_2^2+\beta^2 x_3^2)^{-3/2} = \frac{-P_0\beta x_1}{2\pi R^3} \tag{83}$$

Values for $I_w(r, z, \beta)$ calculated for varying r/z values are presented in Table 1 for μ=0. Values of $I_w(r, z, \beta)$ are similarly calculated for varying values of r/z and shown in Table 2 for values of Poisson's ratio μ=0 and μ=0.40.

Values of the dimensionless vertical stress influence factors for the Westergaard problem, $I_w(r, z, \beta = \sqrt{0.5})$ for varying values of r/z are determined and shown in Table 1. Table of values of the non-dimensional vertical stress influence coefficient for the Westergaard half-space for μ=0 for varying r/z is given in Table 1. The results are identical with results previously presented by Fadum (1948). Table 1 presents identical results obtained and presented by Ike (2019a) who used the Hankel (Fourier-Bessel) transform method for the same problem. The vertical stress influence factors are also determined for varying Poisson's ratio μ and r/z and shown in Table 2. This table similarly presents identical solutions obtained by Ike (2019a) using the Hankel (Fourier-Bessel) transform method to solve the problem.

Solution for the Case of Finite Line Load with Uniform Intensity q_0

The case of a uniformly distributed finite length line load of intensity q_0 on a Westergaard half-space illustrated in Figure 2 was considered. The vertical stress field for a finite length line load of intensity q_0 is obtained by using the solution for a vertical concentrated force at a reference point on the boundary as a Green function, and evaluation of the resulting integral over the finite length of the loaded line as:

$$\sigma_{33}(x_1, x_2, x_3) = \int_0^L -\frac{1}{x_3^2}\frac{\beta}{2\pi}\left(\beta^2 + \frac{x_1^2+x_2^2}{x_3^2}\right)^{-3/2} q_0\, dx_2 \tag{84}$$

$$x_1 = B \text{ and, } 0 \le x_2 \le L \tag{85}$$

Table 1. Influence coefficients for finding spatial variation of vertical stresses caused by vertical concentrated force applied at a reference point on the horizontal boundary of a Westergaard half-space

r/z	I_w	***r/z***	I_w	***r/z***	I_w	***r/z***	I_w	***r/z***	I_w
0.00	**0.3183**	**0.50**	**0.1733**	**1.00**	**0.0613**	**1.50**	**0.0247**	**2.00**	**0.0118**
0.01	0.3182	0.51	0.1698	1.01	0.0601	1.51	0.0243	2.01	0.0116
0.02	0.3179	0.52	0.1664	1.02	0.0589	1.52	0.0239	2.02	0.0115
0.03	0.3175	0.53	0.1631	1.03	0.0577	1.53	0.0235	2.03	0.0113
0.04	0.3168	0.54	0.1598	1.04	0.0566	1.54	0.0231	2.04	0.0112
0.05	0.3159	0.55	0.1566	1.05	0.0555	1.55	0.0228	2.05	0.0110
0.06	0.3149	0.56	0.1534	1.06	0.0544	1.56	0.0224	2.06	0.0109
0.07	0.3137	0.57	0.1502	1.07	0.0534	1.57	0.0220	2.07	0.0108
0.08	0.3123	0.58	0.1471	1.08	0.0523	1.58	0.0217	2.08	0.0106
0.09	0.3107	0.59	0.1441	1.09	0.0513	1.59	0.0214	2.09	0.0105
0.10	**0.3090**	**0.60**	**0.1411**	**1.10**	**0.0503**	**1.60**	**0.0210**	**2.10**	**0.0103**
0.11	0.3071	0.61	0.1382	1.11	0.0494	1.61	0.0207	2.11	0.0102
0.12	0.3050	0.62	0.1353	1.12	0.0484	1.62	0.0204	2.12	0.0101
0.13	0.3028	0.63	0.1325	1.13	0.0475	1.63	0.0201	2.13	0.0100
0.14	0.3005	0.64	0.1297	1.14	0.0466	1.64	0.0198	2.14	0.0098
0.15	0.2980	0.65	0.1270	1.15	0.0458	1.65	0.0195	2.15	0.0097
0.16	0.2953	0.66	0.1244	1.16	0.0449	1.66	0.0192	2.16	0.0096
0.17	0.2926	0.67	0.1218	1.17	0.0441	1.67	0.0189	2.17	0.0095
0.18	0.2897	0.68	0.1192	1.18	0.0432	1.68	0.0186	2.18	0.0094
0.19	0.2867	0.69	0.1167	1.19	0.0424	1.69	0.0183	2.19	0.0092
0.20	**0.2836**	**0.70**	**0.1143**	**1.20**	**0.0417**	**1.70**	**0.0180**	**2.20**	**0.0091**
0.21	0.2804	0.71	0.1119	1.21	0.0409	1.71	0.0178	2.21	0.0090
0.22	0.2771	0.72	0.1095	1.22	0.0401	1.72	0.0175	2.22	0.0089
0.23	0.2737	0.73	0.1072	1.23	0.0394	1.73	0.0172	2.23	0.0088
0.24	0.2703	0.74	0.1050	1.24	0.0387	1.74	0.0170	2.24	0.0087
0.25	0.2668	0.75	0.1028	1.25	0.0380	1.75	0.0167	2.25	0.0086
0.26	0.2632	0.76	0.1006	1.26	0.0373	1.76	0.0165	2.26	0.0085
0.27	0.2595	0.77	0.0985	1.27	0.0366	1.77	0.0163	2.27	0.0084
0.28	0.2558	0.78	0.0964	1.28	0.0360	1.78	0.0160	2.28	0.0083
0.29	0.2521	0.79	0.0944	1.29	0.0354	1.79	0.0158	2.29	0.0082
0.30	**0.2483**	**0.80**	**0.0925**	**1.30**	**0.0347**	**1.80**	**0.0156**	**2.30**	**0.0081**
0.31	0.2445	0.81	0.0905	1.31	0.0341	1.81	0.0153	2.31	0.0080
0.32	0.2407	0.82	0.0887	1.32	0.0335	1.82	0.0151	2.32	0.0079
0.33	0.2369	0.83	0.0868	1.33	0.0329	1.83	0.0149	2.33	0.0078
0.34	0.2330	0.84	0.0850	1.34	0.0324	1.84	0.0147	2.34	0.0077
0.35	0.2291	0.85	0.0833	1.35	0.0318	1.85	0.0145	2.35	0.0076
0.36	0.2253	0.86	0.0815	1.36	0.0313	1.86	0.0143	2.36	0.0075
0.37	0.2214	0.87	0.0799	1.37	0.0307	1.87	0.0141	2.37	0.0074
0.38	0.2176	0.88	0.0782	1.38	0.0302	1.88	0.0139	2.38	0.0074
0.39	0.2137	0.89	0.0766	1.39	0.0297	1.89	0.0137	2.39	0.0073
0.40	**0.2099**	**0.90**	**0.0751**	**1.40**	**0.0292**	**1.90**	**0.0135**	**2.40**	**0.0072**
0.41	0.2061	0.91	0.0735	1.41	0.0287	1.91	0.0133	2.41	0.0071
0.42	0.2023	0.92	0.0720	1.42	0.0282	1.92	0.0131	2.42	0.0070
0.43	0.1986	0.93	0.0706	1.43	0.0277	1.93	0.0130	2.43	0.0069
0.44	0.1948	0.94	0.0692	1.44	0.0273	1.94	0.0128	2.44	0.0069
0.45	0.1911	0.95	0.0678	1.45	0.0268	1.95	0.0126	2.45	0.0068
0.46	0.1875	0.96	0.0664	1.46	0.0264	1.96	0.0124	2.46	0.0067
0.47	0.1839	0.97	0.0651	1.47	0.0259	1.97	0.0123	2.47	0.0066
0.48	0.1803	0.98	0.0638	1.48	0.0255	1.98	0.0121	2.48	0.0066
0.49	0.1768	0.99	0.0665	1.49	0.0251	1.99	0.0120	2.49	0.0065
2.50	**0.0064**	**2.70**	**0.0052**	**2.90**	**0.0042**	**3.20**	**0.0032**	**4.00**	**0.0017**
2.51	0.0064	2.71	0.0051	2.91	0.0042	3.22	0.0031	4.10	0.0016
2.52	0.0063	2.72	0.0051	2.92	0.0042	3.24	0.0031	4.20	0.0015

2.53	0.0062	2.73	0.0050	2.93	0.0041	3.26	0.0030	4.30	0.0014
2.54	0.0061	2.74	0.0050	2.94	0.0041	3.28	0.0030	4.40	0.0013
2.55	0.0061	2.75	0.0049	2.95	0.0040	3.30	0.0029	4.50	0.0012
2.56	0.0060	2.76	0.0049	2.96	0.0040	3.32	0.0029	4.60	0.0011
2.57	0.0059	2.77	0.0048	2.97	0.0040	3.34	0.0028	4.70	0.0011
2.58	0.0059	2.78	0.0048	2.98	0.0039	3.36	0.0028	4.80	0.0010
2.59	0.0058	2.79	0.0047	2.99	0.0039	3.38	0.0027	4.90	0.0009
2.60	**0.0058**	**2.80**	**0.0047**	**3.00**	**0.0038**	**3.40**	**0.0027**	**5.00**	**0.0009**
2.61	0.0057	2.81	0.0046	3.02	0.0038	3.42	0.0026	5.20	0.0008
2.62	0.0056	2.82	0.0046	3.04	0.0037	3.44	0.0026	5.40	0.0007
2.63	0.0056	2.83	0.0045	3.06	0.0036	3.46	0.0026	5.50	0.0007
2.64	0.0055	2.84	0.0045	3.08	0.0036	3.48	0.0025	5.60	0.0006
2.65	0.0054	2.85	0.0045	3.10	0.0035	3.50	0.0025	5.80	0.0006
2.66	0.0054	2.86	0.0044	3.12	0.0034	3.52	0.0023		
2.67	0.0053	2.87	0.0044	3.14	0.0034	3.54	0.0021	**6.00**	**0.0005**
2.68	0.0053	2.88	0.0043	3.16	0.0033	3.56	0.0020	**7.00**	**0.0003**
2.69	0.0052	2.89	0.0043	3.18	0.0033	3.58	0.0018	**8.00**	**0.0002**
								9.00	**0.0002**
								10.00	**0.0001**
								∞	**0.0000**

Note: Solution obtained by cosine integral transform method, present work. This is identical with results obtained by Ike (2019a) using Hankel transformation method $\sigma_{33}(r, x_3) = \frac{-P_0 I_w}{z^2}(r, z, \beta(\mu)) = \frac{P_0 I_w}{x_3^2}$ (for $\mu = 0$).

Table 2. Westergaard vertical stress influence coefficient for point load applied at the origin on the surface of an elastic half-space for various values of Poisson ratio μ and (*r*/*z*).

r*/*z	***μ* = 0**	***μ* = 0.4**
0	0.3183	0.9549
0.1	0.3090	0.8750
0.2	0.2836	0.6916
0.5	0.1733	0.2416
0.8	0.0925	0.0897
1.0	0.0613	0.0516
1.5	0.0247	0.0173
2	0.0118	0.0076
2.5	0.0064	0.0040
3	0.0038	0.0023
3.5	0.0025	0.0015
4	0.0017	0.0010

Note: This results is identical with solutions given by Ike (2019a) using the Hankel transformation method.

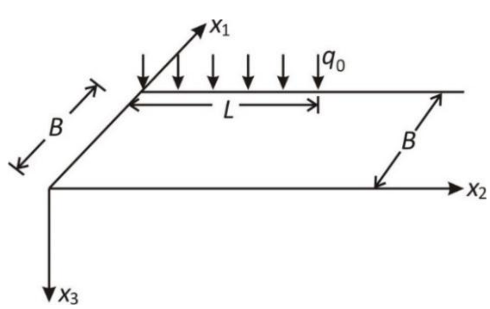

Fig. 2. Uniformly distributed line load of finite length and intensity q_0 acting on the surface of a Westergaard half-space

$$\sigma_{33}(x_1, x_2, x_3) = \frac{-q_0\beta}{2\pi x_3^2}\int_0^L \left(\beta^2 + \frac{B^2}{x_3^2} + \frac{x_2^2}{x_3^2}\right) dx_2 \quad (86)$$

Integration yields:

$$\sigma_{33}(x_1, x_2, x_3) = \frac{-q_0}{x_3^2}\frac{\beta}{2\pi}\left(\frac{n}{(m^2+\beta^2)} \times \frac{1}{(m^2+n^2+\beta^2)^{1/2}}\right) \quad (87)$$

Then,

where $m = \dfrac{B}{x_3}$ (88)

$$n = \frac{L}{x_3} \quad (89)$$

So, $\sigma_{33}(x_1, x_2, x_3) = \dfrac{-q_0}{x_3^2} I(\beta, m, n)$ (90)

where,

$$I(\beta, m, n) = \frac{\beta}{2\pi}\left(\frac{n}{(m^2+\beta^2)(m^2+n^2+\beta^2)^{1/2}}\right) \quad (91)$$

Spatial Distribution of Vertical Stresses under a Corner for a Rectangular Area on the Westergaard Half-Space: Case of Distributed Loading with Uniform Intensity

Here the rectangular area considered lies on the surface of a Westergaard half-space as illustrated in Figure 3.

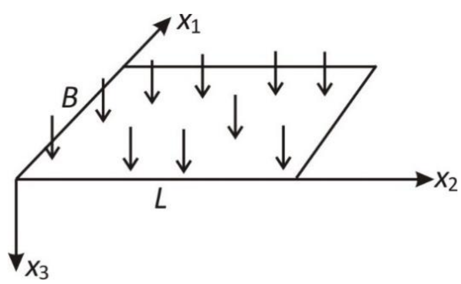

Fig. 3. Uniformly distributed load on a rectangular area on the Westergaard half-space

The spatial variation of vertical stresses in a Westergaard half-space problem caused by uniform distribution of loading with intensity, p_0 on the boundary surface is obtained by using the vertical concentrated force solution as a Kernel function thus:

$$\sigma_{33}(x_1, x_2, x_3) = -\int_0^L\int_0^B \frac{p_0 dx_1 dx_2}{x_3^2} \cdot \frac{\beta}{2\pi}\left(\beta^2 + \frac{x_1^2 + x_2^2}{x_3^2}\right)^{-3/2} \quad (92)$$

Factoring out the constants, we have:

$$\frac{-p_0\beta}{2\pi x_3^2}\int_0^L\int_0^B \left(\frac{\beta^2 x_3^2 + x_1^2 + x_2^2}{x_3^2}\right)^{-3/2} dx_1 dx_2 \quad (93)$$

Integrating, we have:

$$\sigma_{33}(x_1, x_2, x_3) = \frac{-p_0}{2\pi}\tan^{-1}\left(\frac{LB}{\beta x_3}\frac{1}{(L^2+B^2+\beta^2 x_3^2)^{1/2}}\right) \quad (94)$$

The vertical stress is presented in non-dimensional values, m and n defined as:

Let $m = \dfrac{B}{x_3}$, $n = \dfrac{L}{x_3}$, then

$$\sigma_{33}(x_1, x_2, x_3) = \frac{-p_0}{2\pi}\tan^{-1}$$

$$\left(\frac{L}{x_3}\frac{B}{x_3}\frac{x_3^2}{\beta x_3}\left(\frac{1}{\left(\left(\left(\left(\frac{L}{x_3}\right)^2 + \left(\frac{B}{x_3}\right)^2 + \frac{\beta^2 x_3^2}{x_3^2}\right)x_3^2\right)^{1/2}}\right)\right) \quad (95)$$

Alternatively,

$$\sigma_{33}(x_1, x_2, x_3) = \frac{-p_0}{2\pi}\tan^{-1}\left(\frac{mn}{\beta x_3}x_3^2\frac{1}{\left((m^2+n^2+\beta^2)\cdot x_3^2\right)^{1/2}}\right) \quad (96)$$

or,

$$\sigma_{33}(x_1, x_2, x_3) = \frac{-p_0}{2\pi}\tan^{-1}\left(\frac{mn}{\beta x_3}x_3^2\frac{1}{\left(m^2+n^2+\beta^2\right)^{1/2} x_3}\right) \quad (97)$$

After simplifying,

$$\sigma_{33}(x_1, x_2, x_3) = \frac{-p_0}{2\pi}\tan^{-1}\left(\frac{mn}{\beta\left(m^2+n^2+\beta^2\right)^{1/2}}\right) \quad (98)$$

Hence,

$$\sigma_{33}(x_1, x_2, x_3) = p_0 I_w(m, n, \beta) \quad (99)$$

where,

$$I_w(m, n, \beta) = \frac{1}{2\pi}\tan^{-1}\left(\frac{mn}{\beta(m^2+n^2+\beta^2)^{1/2}}\right) \quad (100)$$

The vertical stress influence coefficient $I_w(m, n, \beta)$ obtained from Equation (100) for $\mu = 0, \quad \beta = 0.5$ for uniform distribution of loading on the area $L \times B$ on Westergaard half-space for points at arbitrary depths x_3 below the corners of the rectangular area are presented in Table 3.

Solution for Vertical Stress Field under the Centre of Uniformly Loaded Circular Area on Westergaard Half-Space

A distributed loading of uniform intensity, p over a circular area lying on the Westergaard half-space, illustrated in Figure 4 is similarly considered.

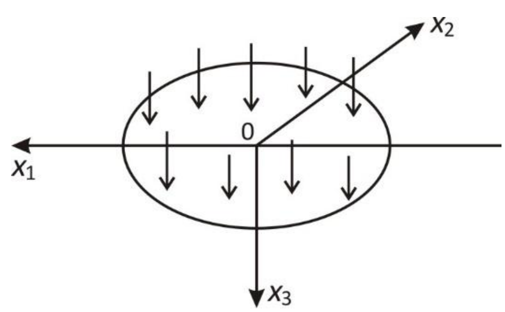

Fig. 4. Uniformly distributed load on a circular area lying on the surface of Westergaard half-space

Using the solution for vertical stress field due to a vertical concentrated loading applied at a reference point of a Westergaard half-space as Green function, the vertical stress field under the centre of a circular area on the Westergaard half-space is expressed as follows:

$$\sigma_{33}(x_1, x_2, x_3) = -\iint_{R_*^2} \frac{\beta}{2\pi x_3^2}\left(\beta^2 + \frac{x_1^2+x_2^2}{x_3^2}\right)^{-3/2} p\, dx_1 dx_2 \quad (101)$$

where R_*^2: is the two dimensional domain of the circular area given by the limits on the r and θ variables: $0 \le r \le R_1, \quad 0 \le \theta \le 2\pi$; R_1 is the radius of the circular area.

Then,

$$\sigma_{33} = -\int_0^{R_1}\int_0^{2\pi} \frac{\beta}{2\pi x_3^2}\left(\frac{\beta^2 x_3^2 + x_1^2 + x_2^2}{x_3^2}\right)^{-3/2} p\, dx_1 dx_2 \quad (102)$$

This integration in Eq. (102) is more conveniently evaluated by transformation to the cylindrical polar coordinates system. The transformation equations from the 3D Cartesian to the cylindrical polar coordinates system are given by Eqs. (55-57).

Then,

$$\sigma_{zz}(r, z) = -\int_0^{R_1}\int_0^{2\pi} \frac{\beta}{2\pi z^2}\left(\beta^2 + \frac{r^2}{z^2}\right)^{-3/2} p|J|\, dr d\theta \quad (103)$$

where $r^2 = x_1^2 + x_2^2$ (104)

and $|J|$: denotes the Jacobian of the transformation from Cartesian coordinates to cylindrical polar coordinates system.
Hence,

$$\sigma_{zz}(r, z) = -\int_0^{R_1}\int_0^{2\pi} \frac{\beta}{2\pi z^2}\left(\beta^2 + \frac{r^2}{z^2}\right)^{-3/2} pr\, dr d\theta \quad (105)$$

After rearranging,

$$\sigma_{zz}(r, z) = -p\int_0^{2\pi} d\theta \int_0^{R_1} \frac{\beta}{2\pi z^2}\left(\beta^2 + \frac{r^2}{z^2}\right)^{-3/2} r dr \quad (106)$$

By further simplifying,

$$\sigma_{zz}(r, z) = \frac{-2\pi p\beta}{2\pi z^2}\int_0^{R_1}\left(\beta^2 + \frac{r^2}{z^2}\right)^{-3/2} r dr \quad (107)$$

Hence,

$$\sigma_{zz}(r,z) = \frac{-p\beta}{z^2}\int_0^{R_1}\left(\beta^2 + \frac{r^2}{z^2}\right)^{-3/2} rdr \quad (108)$$

This simplifies equation to an ordinary integral which is evaluated by the method of substitution of new integration variables $a_1(r)$ defined as:

$$a_1(r) = \beta^2 + \frac{r^2}{z^2} \quad (109)$$

Then $2rdr = z^2 da_1(r)$ (110)

And the problem becomes:

$$\sigma_{zz} = \frac{-p\beta}{z^2}\int_0^{R_1} a_1^{-3/2}\frac{z^2 da_1}{2} = \frac{-p\beta}{2}\int_0^{R_1} a_1^{-3/2} da_1 \quad (111)$$

After simplifying,

$$\sigma_{zz} = -p\beta\left[-\left(\beta^2 + \frac{r^2}{z^2}\right)^{-1/2}\right]_0^{R_1} \quad (112)$$

Substitution of integration limits yields:

$$\sigma_{zz} = -p\left(1 - \left(\frac{\beta^2}{\beta^2 + \frac{R_1^2}{z^2}}\right)^{1/2}\right) = -pI_c(\mu, R_1/z) \quad (113)$$

where,

$$I_c(\mu, R_1/z) = 1 - \sqrt{\left(\frac{\beta^2}{\beta^2 + \left(\frac{R_1}{z}\right)^2}\right)} \quad (114)$$

or,

$$I_c(\mu, R_1/z) = \left\{1 - \sqrt{\left(\frac{\beta^2}{\beta^2 + \left(\frac{D}{2z}\right)^2}\right)}\right\} \quad (115)$$

where D: is the diameter of the circular foundation, $I_c(\mu, R_1/z)$: is the Westergaard vertical stress influence coefficient/factor for vertical stress field at any arbitrary depth z below the center of a circular foundation of radius R_1 carrying uniformly distributed load of known intensity p.

Table 4 which agrees with results presented by Fadum (1948), shows values of tabulated vertical stress influence coefficients $I_c(\mu, R_1/z)$ for various values of R_1/z ranging from 0 to ∞ for points at arbitrary depth z under the centre of a uniformly loaded circular area on a Westergaard half-space, and the corresponding values from an analytical extension of the Boussinesq point load solution. Table 5 presents a comparison of vertical stress influence coefficients for vertical point load applied at the origin of an elastic half-space for both the Boussinesq half-space and the Westergaard half-space models (for μ=0).

DISCUSSION

In this paper, the governing equation of the Westergaard problem has been derived from first principles consideration of the strain – displacement relations, the generalised Hooke's equations, relating stresses to strains and the differential equations of static equilibrium. Disregarding body forces and assuming horizontal inextensibility the governing equation is a partial differential equation expressed mathematically using the vertical displacement by Eq. (36). The governing equation is required to satisfy boundary conditions imposed by deformations and loading.

Table 3. Vertical stress influence coefficients $I_w(m, n, \beta)$ for uniformly distributed load on rectangular foundation area with plan dimensions $L \times B$ on Westergaard half-space (for points at depth x_3 under a corner of the rectangular area for $\mu = 0, \ \beta^2 = 1/2$)

B/x_3 \ L/x_3	0.1	0.2	0.4	0.6	0.8	1.0	2	∞
0.1	0.003	0.006	0.011	0.014	0.017	0.018	0.021	0.022
0.2	0.006	0.012	0.021	0.028	0.033	0.036	0.041	0.044
0.4	0.011	0.021	0.03897	0.0516	0.060	0.066	0.077	0.082
0.6	0.014	0.023	0.0516	0.069	0.081	0.089	0.104	0.112
0.8	0.017	0.033	0.060	0.081	0.095	0.105	0.125	0.135
1.0	0.018	0.036	0.066	0.089	0.105	0.11614	0.140	0.152
2	0.021	0.041	0.077	0.104	0.125	0.140	0.174	0.196
∞	0.022	0.044	0.082	0.112	0.135	0.152	0.196	0.250

Table 4. Vertical stress influence coefficients I_c for vertical stress fields at an arbitrary depth z below the centre of a uniformly loaded circular area on a Westergaard half-space (for μ = 0)

R_1/z	$\hat{I}_B$	I_C	R_1/z	$\hat{I}_B$	I_C	R_1/z	$\hat{I}_B$	I_C	R_1/z	$\hat{I}_B$	I_C
0.00	0.0000	0.0000	0.38	0.1832	0.1191	0.76	0.4953	0.3188	1.14	0.7132	0.4729
0.01	0.0002	0.0001	0.39	0.1913	0.1244	0.77	0.5026	0.3236	1.15	0.7175	0.4762
0.02	0.0006	0.0004	0.40	0.1996	0.1296	0.78	0.5098	0.3284	1.16	0.7216	0.4795
0.03	0.0014	0.0009	0.41	0.2070	0.1349	0.79	0.5169	0.3331	1.17	0.7257	0.4828
0.04	0.0024	0.0016	0.42	0.2163	0.1402	0.80	0.5239	0.3377	1.18	0.7298	0.4860
0.05	0.0037	0.0025	0.43	0.2247	0.1456	0.81	0.5308	0.3424	1.19	0.7337	0.4892
0.06	0.0054	0.0036	0.44	0.2332	0.1510	0.82	0.5376	0.3470	1.20	0.7376	0.4923
0.07	0.0073	0.0049	0.45	0.2417	0.1564	0.83	0.5444	0.3515	1.21	0.7415	0.4955
0.08	0.0095	0.0063	0.46	0.2502	0.1616	0.84	0.5511	0.3560	1.22	0.7453	0.4985
0.09	0.0120	0.0080	0.47	0.2587	0.1672	0.85	0.5577	0.3605	1.23	0.7490	0.5016
0.10	0.0148	0.0099	0.48	0.2673	0.1726	0.86	0.5642	0.3649	1.24	0.7526	0.5046
0.11	0.0179	0.0119	0.49	0.2759	0.1781	0.87	0.5706	0.3695	1.25	0.7562	0.5076
0.12	0.0212	0.0141	0.50	0.2845	0.1835	0.88	0.5769	0.3736	1.26	0.7598	0.5106
0.13	0.0248	0.0165	0.51	0.2930	0.1890	0.89	0.5832	0.3779	1.27	0.7632	0.5135
0.14	0.0287	0.0190	0.52	0.3016	0.1944	0.90	0.5893	0.3822	1.28	0.7667	0.5165
0.15	0.0328	0.0218	0.53	0.3102	0.1998	0.91	0.5954	0.3864	1.29	0.7700	0.5193
0.16	0.0372	0.0247	0.54	0.3188	0.2053	0.92	0.6014	0.3906	1.30	0.7733	0.5222
0.17	0.0418	0.0277	0.55	0.3273	0.2107	0.93	0.6073	0.3948	1.31	0.7766	0.5250
0.18	0.0467	0.0309	0.56	0.3358	0.2161	0.94	0.6132	0.3989	1.32	0.7798	0.5278
0.19	0.0518	0.0343	0.57	0.3443	0.2215	0.95	0.6189	0.4029	1.33	0.7830	0.5306
0.20	0.0571	0.0378	0.58	0.3527	0.2268	0.96	0.6246	0.4069	1.34	0.7861	0.5333
0.21	0.0627	0.0414	0.59	0.3611	0.2322	0.97	0.6302	0.4109	1.35	0.7891	0.5360
0.22	0.0684	0.0452	0.60	0.3695	0.2375	0.98	0.6357	0.4149	1.36	0.7921	0.5387
0.23	0.0744	0.0490	0.61	0.3778	0.2428	0.99	0.6411	0.4188	1.37	0.7951	0.5414
0.24	0.0806	0.0531	0.62	0.3861	0.2481	1.00	0.6465	0.4227	1.38	0.7980	0.5440
0.25	0.0869	0.0572	0.63	0.3943	0.2534	1.01	0.6517	0.4265	1.39	0.8008	0.5466
0.26	0.0935	0.0614	0.64	0.4025	0.2586	1.02	0.6569	0.4303	1.40	0.8036	0.5492
0.27	0.1002	0.0658	0.65	0.4106	0.2638	1.03	0.6620	0.4340	1.41	0.8064	0.5517
0.28	0.1070	0.0702	0.66	0.4186	0.2690	1.04	0.6670	0.4377	1.42	0.8091	0.5543
0.29	0.1141	0.0748	0.67	0.4266	0.2741	1.05	0.6720	0.4414	1.43	0.8118	0.5568
0.30	0.1213	0.0794	0.68	0.4345	0.2792	1.06	0.6769	0.4451	1.44	0.8144	0.5592
0.31	0.1286	0.0842	0.69	0.4424	0.2843	1.07	0.6817	0.4487	1.45	0.8170	0.5617
0.32	0.1361	0.0890	0.70	0.4502	0.2893	1.08	0.6864	0.4522	1.46	0.8196	0.5641
0.33	0.1436	0.0938	0.71	0.4579	0.2943	1.09	0.6910	0.4558	1.47	0.8221	0.5665
0.34	0.1513	0.0988	0.72	0.4655	0.2995	1.10	0.6956	0.4593	1.48	0.8245	0.5689
0.35	0.1592	0.1038	0.73	0.4731	0.3043	1.11	0.7001	0.4627	1.49	0.8269	0.5713
0.36	0.1671	0.1089	0.74	0.4806	0.3091	1.12	0.7046	0.4662	1.50	0.8293	0.5736

0.37	0.1751	0.1140	0.75	0.4880	0.3140	1.13	0.7089	0.4695	1.51	0.8317	0.5759
1.52	0.8340	0.5782	1.79	0.8840	0.6326	2.15	0.9250	0.6876	4.00	0.9857	0.8259
1.53	0.8362	0.5805	1.80	0.8855	0.6344	2.20	0.9291	0.6940	4.20	0.9876	0.8340
1.54	0.8385	0.5827	1.81	0.8869	0.6361	2.25	0.9330	0.7002	4.40	0.9891	0.8413
1.55	0.8407	0.5850	1.82	0.8883	0.6379	2.30	0.9366	0.7061	4.60	0.9904	0.8481
1.56	0.8428	0.5872	1.83	0.8897	0.6396	2.35	0.9400	0.7119	4.80	0.9915	0.8543
1.57	0.8450	0.5893	1.84	0.8911	0.6413	2.40	0.9431	0.7174	5.00	0.9925	0.8600
1.58	0.8470	0.5915	1.85	0.8925	0.6430	2.45	0.9460	0.7227	5.20	0.9933	0.8653
1.59	0.8491	0.5937	1.86	0.8938	0.6447	2.50	0.9488	0.7278	5.40	0.9940	0.8702
1.60	0.8511	0.5958	1.87	0.8951	0.6463	2.55	0.9513	0.7328	5.60	0.9946	0.8747
1.61	0.8531	0.5979	1.88	0.8964	0.6480	2.60	0.9537	0.7376	5.80	0.9951	0.8790
1.62	0.8551	0.6000	1.89	0.8977	0.6496	2.65	0.9560	0.7422	6.00	0.9956	0.8830
1.63	0.8570	0.6020	1.90	0.8990	0.6512	2.70	0.9581	0.7467	6.50	0.9965	0.8919
1.64	0.8589	0.6041	1.91	0.9002	0.6528	2.75	0.9601	0.7510	7.00	0.9972	0.8995
1.65	0.8608	0.6061	1.92	0.9014	0.6544	2.80	0.9620	0.7552	7.50	0.9977	0.9061
1.66	0.8626	0.6081	1.93	0.9026	0.6560	2.85	0.9637	0.7592	8.00	0.9981	0.9120
1.67	0.8644	0.6101	1.94	0.9038	0.6576	2.90	0.9654	0.7631	9.00	0.9987	0.9217
1.68	0.8662	0.6121	1.95	0.9050	0.6591	2.95	0.9669	0.7669	10.00	0.9990	0.9295
1.69	0.8679	0.6140	1.96	0.9061	0.6606	3.00	0.9684	0.7706	12.00	0.9994	0.9412
1.70	0.8697	0.6160	1.97	0.9073	0.6622	3.10	0.9711	0.7776	14.00	0.9996	0.9496
1.71	0.8714	0.6179	1.98	0.9084	0.6637	3.20	0.9735	0.7842	16.00	0.9998	0.9559
1.72	0.8730	0.6198	1.99	0.9095	0.6652	3.30	0.9756	0.7905	18.00	0.9998	0.9608
1.73	0.8747	0.6217	2.00	0.9106	0.6667	3.40	0.9775	0.7964	20.00	0.9999	0.9647
1.74	0.8763	0.6235	2.02	0.9127	0.6696	3.50	0.9793	0.8020	25.00	0.9999	0.9717
1.75	0.8779	0.6254	2.04	0.9147	0.6725	3.60	0.9808	0.8073	30.00	1.0000	0.9764
1.76	0.8794	0.6272	2.06	0.9167	0.6753	3.70	0.9822	0.8123	40.00	1.0000	0.9823
1.77	0.8810	0.6290	2.08	0.9187	0.6781	3.80	0.9835	0.8171	50.00	1.0000	0.9859
1.78	0.8825	0.6308	2.10	0.9205	0.6809	3.90	0.9847	0.8216	100.00	1.0000	0.9929
									∞	1.0000	1.0000

Note: R_1: is radius of the circular area.

Table 5. Vertical stress influence coefficients for vertical normal stresses due to vertical point load applied at the origin of an elastic half-space, Boussinesq half-space: $\sigma_{33} = \frac{P}{x_3^2} I_B$ and Westergaard half-space (μ = 0): $\sigma_{33} = \frac{P}{x_3^2} I_W$

r/z	I_B	I_W	***r/z***	I_B	I_W	***r/z***	I_B	I_W	***r/z***	I_B	I_W
0.00	0.4775	0.3183	0.16	0.4482	0.2953	0.32	0.3742	0.2407	0.48	0.2843	0.1803
0.01	0.4773	0.3182	0.17	0.4446	0.2926	0.33	0.3687	0.2369	0.49	0.2788	0.1768
0.02	0.4770	0.3179	0.18	0.4409	0.2897	0.34	0.3632	0.2330	0.50	0.2733	0.1733
0.03	0.4764	0.3175	0.19	0.4370	0.2867	0.35	0.3577	0.2291	0.51	0.2679	0.1698
0.04	0.4756	0.3168	0.20	0.4329	0.2836	0.36	0.3521	0.2253	0.52	0.2625	0.1664
0.05	0.4745	0.3159	0.21	0.4286	0.2804	0.37	0.3465	0.2214	0.53	0.2571	0.1631
0.06	0.4732	0.3149	0.22	0.4242	0.2771	0.38	0.3408	0.2176	0.54	0.2518	0.1598
0.07	0.4717	0.3137	0.23	0.4197	0.2737	0.39	0.3351	0.2137	0.55	0.2466	0.1566
0.08	0.4699	0.3123	0.24	0.4151	0.2703	0.40	0.3294	0.2099	0.56	0.2414	0.1534
0.09	0.4679	0.3107	0.25	0.4103	0.2668	0.41	0.3238	0.2061	0.57	0.2363	0.1502
0.10	0.4657	0.3090	0.26	0.4054	0.2632	0.42	0.3181	0.2023	0.58	0.2313	0.1471
0.11	0.4633	0.3071	0.27	0.4004	0.2595	0.43	0.3124	0.1986	0.59	0.2263	0.1441
0.12	0.4607	0.3050	0.28	0.3954	0.2558	0.44	0.3068	0.1948	0.60	0.2214	0.1411
0.13	0.4579	0.3028	0.29	0.3902	0.2521	0.45	0.3011	0.1911	0.61	0.2165	0.1382
0.14	0.4548	0.3005	0.30	0.3849	0.2483	0.46	0.2955	0.1875	0.62	0.2117	0.1353
0.15	0.4516	0.2980	0.31	0.3796	0.2445	0.47	0.2899	0.1839	0.63	0.2070	0.1325
0.64	0.2024	0.1297	1.16	0.0567	0.0449	1.68	0.0167	0.0186	2.20	0.0058	0.0091
0.65	0.1978	0.1270	1.17	0.0553	0.0441	1.69	0.0163	0.0183	2.21	0.0057	0.0090
0.66	0.1934	0.1244	1.18	0.0539	0.0432	1.70	0.0160	0.0180	2.22	0.0056	0.0089
0.67	0.1889	0.1218	1.19	0.0526	0.0424	1.71	0.0157	0.0178	2.23	0.0055	0.0088

0.68	0.1864	0.1192	1.20	0.0513	0.0417	1.72	0.0153	0.0175	2.24	0.0054	0.0087
0.69	0.1804	0.1167	1.21	0.0501	0.0409	1.73	0.0150	0.0172	2.25	0.0053	0.0086
0.70	0.1762	0.1143	1.22	0.0489	0.0401	1.74	0.0147	0.0170	2.26	0.0052	0.0085
0.71	0.1721	0.1119	1.23	0.0477	0.0394	1.75	0.0144	0.0167	2.27	0.0051	0.0084
0.72	0.1681	0.1095	1.24	0.0466	0.0387	1.76	0.0141	0.0165	2.28	0.0050	0.0083
0.73	0.1641	0.1072	1.25	0.0454	0.0380	1.77	0.0138	0.0163	2.29	0.0049	0.0082
0.74	0.1603	0.1050	1.26	0.0443	0.0373	1.78	0.0135	0.0160	2.30	0.0048	0.0081
0.75	0.1565	0.1028	1.27	0.0433	0.0366	1.79	0.0132	0.0158	2.31	0.0047	0.0080
0.76	0.1527	0.1006	1.28	0.0422	0.0360	1.80	0.0129	0.0156	2.32	0.0047	0.0079
0.77	0.1491	0.0985	1.29	0.0412	0.0354	1.81	0.0126	0.0153	2.33	0.0046	0.0078
0.78	0.1455	0.0964	1.30	0.0402	0.0347	1.82	0.0124	0.0151	2.34	0.0045	0.0077
0.79	0.1420	0.0944	1.31	0.0393	0.0341	1.83	0.0121	0.0149	2.35	0.0044	0.0076
0.80	0.1386	0.0925	1.32	0.0384	0.0335	1.84	0.0119	0.0147	2.36	0.0043	0.0075
0.81	0.1353	0.0905	1.33	0.0374	0.0329	1.85	0.0116	0.0145	2.37	0.0043	0.0074
0.82	0.1320	0.0887	1.34	0.0365	0.0324	1.86	0.0114	0.0143	2.38	0.0042	0.0074
0.83	0.1288	0.0868	1.35	0.0357	0.0318	1.87	0.0112	0.0141	2.39	0.0041	0.0073
0.84	0.1257	0.0850	1.36	0.0348	0.0313	1.88	0.0109	0.0139	2.40	0.0040	0.0072
0.85	0.1226	0.0833	1.37	0.0340	0.0307	1.89	0.0107	0.0137	2.41	0.0040	0.0071
0.86	0.1196	0.0815	1.38	0.0332	0.0302	1.90	0.0105	0.0135	2.42	0.0039	0.0070
0.87	0.1166	0.0799	1.39	0.0324	0.0297	1.91	0.0103	0.0133	2.43	0.0038	0.0069
0.88	0.1138	0.0782	1.40	0.0317	0.0292	1.92	0.0101	0.0131	2.44	0.0038	0.0069
0.89	0.1110	0.0766	1.41	0.0309	0.0287	1.93	0.0099	0.0130	2.45	0.0037	0.0068
0.90	0.1083	0.0751	1.42	0.0302	0.0282	1.94	0.0097	0.0128	2.46	0.0036	0.0067
0.91	0.1057	0.0735	1.43	0.0295	0.0277	1.95	0.0095	0.0126	2.47	0.0036	0.0066
0.92	0.1031	0.0720	1.44	0.0288	0.0273	1.96	0.0093	0.0124	2.48	0.0035	0.0066
0.93	0.1005	0.0706	1.45	0.0282	0.0268	1.97	0.0091	0.0123	2.49	0.0034	0.0065
0.94	0.0981	0.0692	1.46	0.0275	0.0264	1.98	0.0089	0.0121	2.50	0.0034	0.0064
0.95	0.0956	0.0678	1.47	0.0269	0.0259	1.99	0.0087	0.0120	2.51	0.0033	0.0064
0.96	0.0933	0.0664	1.48	0.0263	0.0255	2.00	0.0085	0.0118	2.52	0.0033	0.0063
0.97	0.0910	0.0651	1.49	0.0257	0.0251	2.01	0.0084	0.0116	2.53	0.0032	0.0062
0.98	0.0887	0.0638	1.50	0.0251	0.0247	2.02	0.0082	0.0115	2.54	0.0032	0.0061
0.99	0.0865	0.0625	1.51	0.0245	0.0243	2.03	0.0081	0.0113	2.55	0.0031	0.0061
1.00	0.0844	0.0613	1.52	0.0240	0.0239	2.04	0.0079	0.0112	2.56	0.0031	0.0060
1.01	0.0823	0.0601	1.53	0.0234	0.0235	2.05	0.0078	0.0110	2.57	0.0030	0.0059
1.02	0.0803	0.0589	1.54	0.0229	0.0231	2.06	0.0076	0.0109	2.58	0.0030	0.0059
1.03	0.0783	0.0577	1.55	0.0224	0.0228	2.07	0.0075	0.0108	2.59	0.0029	0.0058
1.04	0.0764	0.0566	1.56	0.0219	0.0224	2.08	0.0073	0.0106	2.60	0.0029	0.0058
1.05	0.0744	0.0555	1.57	0.0214	0.0220	2.09	0.0072	0.0105	2.61	0.0028	0.0057
1.06	0.0727	0.0544	1.58	0.0209	0.0217	2.10	0.0070	0.0103	2.62	0.0028	0.0056
1.07	0.0709	0.0534	1.59	0.0204	0.0214	2.11	0.0069	0.0102	2.63	0.0027	0.0056
1.08	0.0691	0.0523	1.60	0.0200	0.0210	2.12	0.0068	0.0101	2.64	0.0027	0.0055
1.09	0.0674	0.0513	1.61	0.0195	0.0207	2.13	0.0066	0.0100	2.65	0.0026	0.0054
1.10	0.0658	0.0503	1.62	0.0191	0.0204	2.14	0.0065	0.0098	2.66	0.0026	0.0054
1.11	0.0641	0.0494	1.63	0.0187	0.0201	2.15	0.0064	0.0097	2.67	0.0025	0.0053
1.12	0.0626	0.0484	1.64	0.0183	0.0198	2.16	0.0063	0.0096	2.68	0.0025	0.0053
1.13	0.0610	0.0475	1.65	0.0179	0.0195	2.17	0.0062	0.0095	2.69	0.0025	0.0052
1.14	0.0595	0.0466	1.66	0.0175	0.0192	2.18	0.0060	0.0094	2.70	0.0024	0.0052
1.15	0.0581	0.0458	1.67	0.0171	0.0189	2.19	0.0059	0.0092	2.71	0.0024	0.0051
2.72	0.0023	0.0051	2.92	0.0017	0.0042	3.24	0.0011	0.0031	4.20	0.0003	0.0015
2.73	0.0023	0.0050	2.93	0.0017	0.0041	3.26	0.0010	0.0030	4.30	0.0003	0.0014
2.74	0.0023	0.0050	2.94	0.0017	0.0041	3.28	0.0010	0.0030	4.40	0.0003	0.0013
2.75	0.0022	0.0049	2.95	0.0016	0.0040	3.30	0.0010	0.0029	4.50	0.0002	0.0012
2.76	0.0022	0.0049	2.96	0.0016	0.0040	3.32	0.0009	0.0029	4.60	0.0002	0.0011
2.77	0.0022	0.0048	2.97	0.0016	0.0040	3.34	0.0009	0.0028	4.70	0.0002	0.0011
2.78	0.0021	0.0048	2.98	0.0016	0.0039	3.36	0.0009	0.0028	4.80	0.0002	0.0010
2.79	0.0021	0.0047	2.99	0.0015	0.0039	3.38	0.0009	0.0027	4.90	0.0002	0.0009

2.80	0.0021	0.0047	3.00	0.0015	0.0038	3.40	0.0009	0.0027	5.00	0.0001	0.0009
2.81	0.0020	0.0046	3.02	0.0015	0.0038	3.42	0.0008	0.0026	5.20	0.0001	0.0008
2.82	0.0020	0.0046	3.04	0.0014	0.0037	3.44	0.0008	0.0026	5.40	0.0001	0.0007
2.83	0.0020	0.0045	3.06	0.0014	0.0036	3.46	0.0008	0.0026	5.50	0.0001	0.0007
2.84	0.0019	0.0045	3.08	0.0013	0.0036	3.48	0.0008	0.0025	5.60	0.0001	0.0006
2.85	0.0019	0.0045	3.10	0.0013	0.0035	3.50	0.0007	0.0025	5.80	0.0001	0.0006
2.86	0.0019	0.0044	3.12	0.0013	0.0034	3.60	0.0007	0.0023	6.00	0.0001	0.0005
2.87	0.0019	0.0044	3.14	0.0012	0.0034	3.70	0.0006	0.0021	7.00	0.0000	0.0003
2.88	0.0018	0.0043	3.16	0.0012	0.0033	3.80	0.0005	0.0020	8.00	0.0000	0.0002
2.89	0.0018	0.0043	3.18	0.0012	0.0033	3.90	0.0005	0.0018	9.00	0.0000	0.0002
2.90	0.0018	0.0042	3.20	0.0011	0.0032	4.00	0.0004	0.0017	10.00	0.0000	0.0001
2.91	0.0017	0.0042	3.22	0.0011	0.0031	4.10	0.0004	0.0016	∞	0.0000	0.0000

Note: Vertical Stress Influence Factors (Coefficients) obtained from present study for the Boussinesq problem.

The cosine integral transform of the domain equation is shown in Eq. (37). The linearity properties of the cosine integral transformation and the Leibnitz rule were used to express the transformed problem as ODEs obtained as Eq. (41) in terms of $\bar{U}_3(\alpha_1, \alpha_2, x_3)$, the cosine integral transform of the vertical displacement $u_3(x_1, x_2, x_3)$.

The differential operator method or trial function method was used to solve the ODE to obtain the general solution for $\bar{U}_3(\alpha_1, \alpha_2, x_3)$ in the cosine integral transform space as the Eq. (42) which contains two unknown integration constants c_1 and c_2. The requirement that the vertical displacement be finite and bounded as the depth tends to infinity is used to obtain one of the constants of integration as Eq. (43), yielding the bounded vertical displacement in the cosine integral transformation parameters as Eq. (44). By inversion the bounded vertical displacement is found for the physical problem as Eq. (48) which contains one unknown integration constant.

The vertical stress field is obtained using the vertical stress – displacement equation as Eq. (50) which contains one integration constant, c_1. The value of c_1 in the expression for both the vertical stress field and the vertical displacement is obtained by using the stress boundary condition that requires equilibrium of the resultant internal vertical stresses and the applied vertical concentrated force, yielding Eq. (51). The integration constant is thus obtained by solving Eq. (51) using integration by the method of change of coordinates as Eq. (67). Thus, $u_3(x_1, x_2, x_3)$ is completely determined as Eq. (68) or (69). The normal and shear stress fields are then completely determined by use of the stress – displacement relations as Eq. (98) or (99) where the expression for dimensionless influence factors for corner points of uniformly loaded rectangular foundation areas is given by Eq. (100). Table of values for the influence factors for finding vertical stress at corner points of rectangular foundation areas under uniform loads on Westergaard half-space are presented in Table 3 for values of the Poisson ratio, μ given by $\mu=0$

The case of uniformly loaded circular foundation of radius R_1 on the Westergaard half-space was similarly considered. The solution for vertical concentrated loading at a reference point of Westergaard half-space was similarly used as a Green (kernel or fundamental) function in expressing the spatial distribution of vertical stresses at depth z under the centre of the foundation as Eq. (102).

Evaluation of the integration problem gives the expression for the vertical stresses under the centre of a uniformly loaded circular foundation as Eq. (113). This equation is in terms of influence factors for finding vertical stresses, Eq. (115), and presented as a function of R_1/z in Table 4.

Table 4 also compares the solution obtained using Westergaard problem with the results obtained from an analytical extension of the Boussinesq's point load solution to the same problem of vertical stress under the centre of circular foundations of same radius R_1. For line load of finite length L and uniform intensity q_0 the vertical stress field is obtained as Eq. (87) while the influence coefficient is given by Eq. (91). Similarly the use of the solution for vertical concentrated loading as kernel function was adopted to express the vertical stress distribution under a corner of a rectangular area $L \times B$ carrying uniformly distributed loading as the integral over the rectangular domain given by Eq. (92).

Evaluation of the double integration problem yields the vertical stress field for points under the corner of uniformly loaded rectangular foundation as Eq. (94). The solution for vertical concentrated force at the reference point on the Westergaard half-space was used as a fundamental (Green) function to obtain the spatial distribution of vertical stresses caused by uniformly distributed line load of finite length L and intensity q_0 as Eq. (84). Evaluation of the integral yields the expression for the vertical stress due to the distributed line load as Eq. (87). This equation is presented using non-dimensional vertical stress influence factors as Eq. (90) where the dimensionless vertical stress influence coefficient for vertical stress field for the case of uniformly distributed line loading with finite length was found as Eq. (91).

CONCLUSIONS

It is concluded that,

i) The elasticity problem involving Westergaard half-space subject to a vertical concentrated load applied at the origin has been presented as a boundary value problem (BVP).

ii) The BVP is described (given) by a partial differential equation expressed in terms of vertical displacement component.

iii) The BVP is obtained by simultaneous consideration of the differential equations of static equilibrium when the body forces are disregarded, the geometric relations of strain, and the generalised Hooke's stress – strain relations.

iv) The cosine integral transform method is suitable for finding solutions to the BVP (Cauchy – Navier equation) for the unknown bounded vertical displacement. The boundary condition requirement of equilibrium of resultant vertical stresses and the applied vertical concentrated loading at the origin has been used to find the integration constant, thus leading to the full determination of the vertical displacement field.

v) The cosine integral transform of the domain equation converted the BVP to an integral equation. The integral equation further simplified to an ODE solvable by differential operator methods, trial function methods, variation of parameters and other mathematical tools.

vi) Inversion of the cosine integral transform solution of the bounded vertical displacement in the cosine integral transform parameters yielded bounded solutions for the vertical displacement in terms of one integration constant.

vii) The use of the stress boundary condition requirement of the equilibrium of the resultant vertical stresses and the applied vertical concentrated force at the origin was adopted for obtaining the integration constant, thus completely determining the vertical displacement field.

viii) The spatial distribution of stresses were determined using the kinematic equations on the displacement obtained and the stress-strain equations.

ix) For the vertical concentrated load on the Westergaard medium problem, the expression for the spatial variation of the vertical displacement is a single valued function of the x_1, x_2, x_3 coordinates in the

Westergaard medium. The expression is singular at the point of application of the vertical concentrated load, and at this singular point, the vertical displacement cannot be determined.

xi) At the origin (0, 0, 0) the vertical displacement field is singular, and thus cannot be defined. $u_3(x_1, x_2, x_3)$ is indeterminate at the origin.

xii) The normal stresses σ_{33}, σ_{11}, σ_{22} at any point in the Westergaard half-space where $x_3=0$ are indeterminate (singular or undefined).

xiii) The value of shear stress σ_{12} on the boundary surface (x_1x_2 plane or $x_3=0$ plane) is zero. At all points in the Westergaard elastic half-space the shear stress σ_{12} is also zero. This is consequent on the foundational assumption of horizontal nonextensibility which implies that $u_1=u_2=0$. The horizontal displacement components in the x_1 and x_2 directions all vanish. Hence the shear strain obtained γ_{12} would consequently be zero following from the kinematic relations of small displacement elasticity for the Westergaard half-space.

xiv) The shear stresses σ_{32} and σ_{31} are indeterminate at the origin of the Westergaard elastic half-space, which is a point of singularity for stresses and displacements

NOTATIONS/NOMENCLATURE

r,θ,z: three dimensional polar coordinates.

x_1, x_2, x_3: three dimensional Cartesian coordinates

(0,0,0): origin of a linear elastic half-space

∞: infinity.

L: length of rectangular foundation

R_1: variable defined in terms of x_1, x_2 and βx_3

B: width of rectangular foundation

R: radius of circular foundation

p_0: intensity of uniformly distributed load on a rectangular area

q_0: intensity of uniformly distributed finite length line load

P_0: magnitude of vertical point load applied at the origin

σ_{zz}: vertical stress

σ_{11}, σ_{22}, σ_{33}: normal stresses

σ_{12}, σ_{21}, σ_{32}, σ_{23}, σ_{31}, σ_{13}: shear stresses

f_1, f_2, f_3: body force components in the x_1, x_2 and x_3 coordinate directions, respectively

T: time

u_1, u_2, u_3: displacement components in the x_1, x_2 and x_3: directions respectively

$u_3(x_1, x_2, x_3)$: vertical displacement

ε_v: volumetric strain

G: shear modulus

E: Young's modulus

λ: Lamé constant or Lamé coefficient

μ: Poisson's ratio

ε_{11}, ε_{22}, ε_{33}: normal strains

γ_{12}, γ_{13}, γ_{23}: shear strains

β: elastic parameter defined in terms of the Poisson's ratio, μ or in terms of shear modulus (G) and Lamé constant λ

α_1, α_2: parameters of the two-dimensional cosine integral transform

$\bar{U}_3(\alpha_1, \alpha_2, x_3)$: two-dimensional cosine integral transform of the vertical displacement $u_3(x_1, x_2, x_3)$

c_1, c_2: constants of integration

$\hat{c}_1, \hat{c}_2$: constants related to c_1 and c_2, respectively

∇^2: Laplacian operator

$|J|$: Jacobian of the coordinate transformation from 3D Cartesian to 3D polar coordinates system

$a_0(r)$: changed variable of integration defined in terms of r and βz

$I_w(r, x_3, \beta(\mu))$ or $I_w(r, z, \beta(\mu))$: Westergaard vertical stress influence coefficient for vertical point load applied at the origin of a Westergaard half-space

k_0: coefficient of lateral stress at rest, defined in terms of μ

m: dimensionless parameter defined in terms of B and x_3

n: dimensionless parameter defined in terms of *L* and x_3

$I_w(m, n, \beta)$: vertical stress influence coefficient for uniformly distributed load on rectangular foundation area with plan dimensions $L \times B$ on Westergaard half-space for points at depth x_3 under a corner of the rectangular area

$I(\beta, m, n)$: vertical stress influence coefficient at $(0,0,x_3)$ for uniformly distributed line load of finite length, *L* and intensity q_0 acting at *B* from the x_2 axis and parallel to the x_2 axis on the surface of a Westergaard half-space

$I_C(\mu, R_1/z)$: Westergaard vertical stress influence coefficient/factor for vertical stress field at any arbitrary depth *z* below the centre of a circular foundation of radius R_1 carrying uniformly distributed load of known intensity, *p*

I_B: Boussinesq's vertical stress influence coefficient/factor for vertical stress at any arbitrary point x_3 or *z* below the Boussinesq half-space due to vertical point load applied at the origin

$\hat{I}_B$: vertical stress influence coefficient / factor for vertical stress at any arbitrary point *z* below the centre of a circular foundation under uniformly distributed load (for Boussinesq theory)

R_*^2: two-dimensional domain of the circular area

R^3: three-dimensional region of elastic half-space

3D: three-dimensional

2D: two-dimensional

ODE: Ordinary Differential Equation

ODEs: Ordinary Differential Equations

BVP: Boundary Value Problem

PDE Partial Differential Equation

$\frac{\partial}{\partial x_1}$: partial differential operator (for partial differentiation with respect to x_1)

$\frac{\partial}{\partial x_2}$: partial differential operator (for partial differentiation with respect to x_2)

$\frac{\partial}{\partial x_3}$: partial differential operator (for partial differentiation with respect to x_3)

$\frac{\partial^2}{\partial x_1^2}, \frac{\partial^2}{\partial x_2^2}, \frac{\partial^2}{\partial x_3^2}$: second order partial differential operators (with respect to x_1, x_2 and x_3 respectively)

$\int$: integral

$\iint$: double integral

$|\ \ |$: determinant

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