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Bending Solution for Simply Supported Annular Plates Using the Indirect Trefftz Boundary Method

Ghannadiasl, A.[1*] and Noorzad, A.[2•]

[1] Assistant Professor, Faculty of Engineering, University of Mohaghegh Ardabili, Ardabil, Iran.
[2] Assistant Professor, School of Civil Engineering, College of Engineering, University of Tehran, Tehran, Iran.

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ABSTRACT: This paper presents the bending analysis of annular plates by the indirect Trefftz boundary approach. The formulation for thin and thick plates is based on the Kirchhoff plate theory and the Reissner plate theory. The governing equations are therefore a fourth-order boundary value problem and a sixth-order boundary value problem, respectively. The Trefftz method employs the complete set of solutions satisfying the governing equation. The main benefit of the Trefftz boundary method is that it does not involve singular integrals because of the properties of its solution basis functions. It can therefore be classified into the regular boundary element method. The present method is simple and efficient in comparison with the other methods. In addition, the boundary conditions can be embedded in this method. Finally, several numerical examples are shown to illustrate the efficiency and simplicity of the current approach.

Keywords: Annular Plates, Indirect Trefftz Method, Kirchhoff Plate Theory, Reissner Plate Theory.

INTRODUCTION

Plates are widely used as classical structural components in civil and mechanical engineering projects. Due to their practical importance, a lot of effort is devoted to the analysis of plates. Recently, this subject has been extensively studied by many researchers such as Abdollahzadeh and Ghobadi (2014), Shahabian et al. (2013), Mirzapour et al. (2012) and Ghasemieh and Shamim (2010).

The Kirchhoff plate theory is the simplest plate theory. The shear deformation in the plate thickness is neglected in this theory. The longitudinal elastic modulus is much higher than the transversal and the shear modulus in the thick and moderately thick plates. Therefore, the use of shear deformation plate theory is recommended for the moderately thick and thick plates. The Reissner model is known as first-order shear deformation theory (FSDT). This model accounts for the shear deformation effect through the thickness of the plate in the simplest way. This approach gives satisfactory results for a wide variety of problems, even for thick and moderately thick plates. Because of its computational

* Corresponding author E-mail: aghannadiasl@uma.ac.ir
• Deceased 23 May 2012

efficiency, this model is applied to large-scale computations of industrial applications (Zenkour, 2003).

This paper presents the bending analysis of annular plates by the indirect Trefftz boundary approach. The Trefftz method can be classified into the category of the boundary-type solution procedures. The problem can be solved by discretization of the boundary alone when the governing equation of the object is a linear homogeneous differential equation. Data generation is therefore much easier than the domain-type solution procedures. Furthermore, the Trefftz Method is common and easier than the boundary element method of singular property (Kita and Kamiya, 1995).

In this approach, the trial functions are expanded in a sequence of linear independent Trefftz functions and a discrete set of unknown coefficients **a**. However, the weighting functions may be chosen in different ways. When the Dirac delta function is used, the method leads to the Trefftz collocation method (TCM). If the Trefftz function is employed as the weight function, the method leads to the Trefftz Galerkin method (TGM) (Jin et al., 1993). Trefftz-based formulations have been studied by several authors, such as Zielinski and Zienkiewicz (1985), Cheung et al. (1989), Zielinski (1995), Pluymers et al. (2007), Lee et al. (2007), Liu (2007a, 2007b), Young et al. (2007), Li et al. (2008), Karaś and Zieliński (2008), Qin and Wang (2008), Chen et al. (2009), Lee and Chen (2009), Lee et al. (2010), Li et al. (2010), Chen et al. (2010a, 2010b), Li et al. (2011), Maciąg (2011), Li et al. (2013), Maciąg and Pawińska (2013), Grysa and Maciejewska (2013), Karageorghis (2013), Kretzschmar et al. (2014), Ku et al. (2015), and Brański and Borkowska (2015).

In section the basic equations based on the Kirchhoff plate theory and the Reissner plate theories are described in detail. Then the complete solutions and complete sets are illustrated. The indirect Trefftz method is explained next and some numerical examples are shown to illustrate the efficiency of the Trefftz method

BASIC EQUATIONS

An annular plate is considered in a Cartesian coordinate system x_i, with its middle surface in the x_1–x_2 plane i.e. $x_3 = 0$ (Figure 1). The material of the annular plate is elastic and isotropic with modulus of elasticity, E, and Poisson's ratio, ν. The load $q(x_1, x_2)$ per square area is applied on the upper surface of the plate, positive in the direction of the axis x_3 (Timoshenko and Woinowsky-Krieger, 1959).

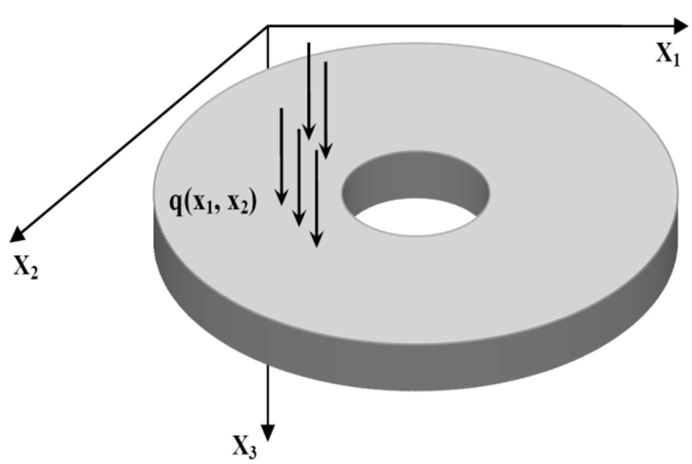

Fig. 1. An annular plate in bending

By assuming small deflections, and in the absence of body forces, the static equilibrium can be as stated as follows (Ghannadiasl and Noorzad, 2007):

$$M_{\alpha\beta,\beta} - Q_\alpha = 0 \tag{1}$$

$$Q_{\alpha,\alpha} + q = 0 \tag{2}$$

where $M_{\alpha\beta}$ and Q_α: are bending moment and shear force per unit length, respectively.

For thin plates, the formulation is based on the Kirchhoff plate theory, so the governing equation is a fourth-order boundary value problem. The formula for thick plates is based on the Reissner plate theory, the governing equation is therefore a sixth-order boundary value problem.

Kirchhoff Plate Theory

According to the Kirchhoff plate theory, the equilibrium equations for axisymmetric bending of the annular plate in question in polar coordinates (Figure 2) can be given by:

$$\frac{\partial Q_r}{\partial r} + \frac{1}{r} Q_r + q = 0 \tag{3}$$

$$\frac{\partial M_r}{\partial r} + \frac{M_r - M_\theta}{r} - Q_r = 0 \tag{4}$$

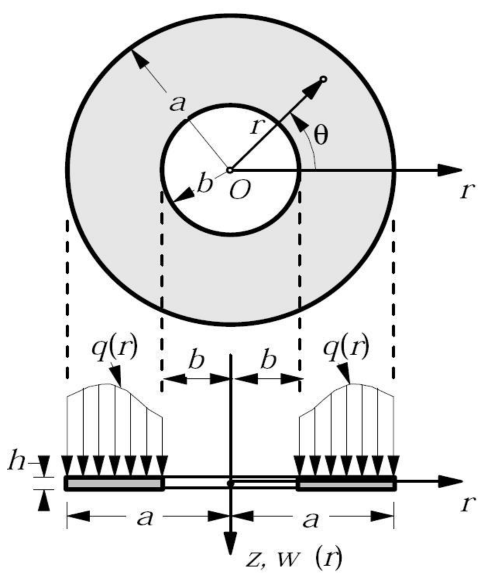

Fig. 2. Reference system and variables

Based on Hooke's law and Kirchhoff's assumptions, the bending moment and displacement relations can be calculated as follows:

$$M_r^K = D \cdot \left[\frac{\partial \varphi_r^K}{\partial r} + \nu \frac{\varphi_r^K}{r} \right] \tag{5}$$

$$M_\theta^K = D \cdot \left[\frac{\varphi_r^K}{r} + \nu \frac{\partial \varphi_r^K}{\partial r} \right] \tag{6}$$

where $D = Eh^3 / 12(1-\nu^2)$: is the flexural rigidity of the plate and φ_r^K : is the angle of rotation from the normal to the mid-surface. The superscript K and w^K: are the Kirchhoff plate quantities and average transverse displacement, respectively. By substituting Eqs. (5) and (6) into Eq. (4), the shear force to displacement relation can be obtained with the following:

$$\begin{aligned} Q_r^K &= D \cdot \left[\frac{\partial^2 \varphi_r^K}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi_r^K}{\partial r} - \frac{\varphi_r^K}{r^2} \right] \\ &= D \cdot \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \cdot \varphi_r^K \right) \right] \end{aligned} \tag{7}$$

Based on the Kirchhoff plate theory, equivalent change of the slopes of the normal on the mid-surface is given as follows:

$$\varphi_r^K = -\frac{\partial w^K}{\partial r} \tag{8}$$

By substituting Eq. (8) into Eqs. (5) and (6), the bending moment to displacement relations can be determined as:

$$M_r^K = -D \cdot \left[\frac{\partial^2 w^K}{\partial r^2} + \frac{\nu}{r} \frac{\partial w^K}{\partial r} \right] \tag{9}$$

$$M_\theta^K = -D \cdot \left[\nu \frac{\partial^2 w^K}{\partial r^2} + \frac{1}{r} \frac{\partial w^K}{\partial r} \right] \tag{10}$$

Also, by substituting Eq. (8) into Eq. (7), the shear force–displacement relation can be obtained by the following:

$$Q_r^K = -D \cdot \frac{\partial}{\partial r}\left(\nabla^2 w^K\right) \tag{11}$$

where ∇: is the Laplace operator. The Marcus moment M^K: is defined as:

$$\mathsf{M}^K = \frac{M_r^K + M_\theta^K}{1+\nu} = -D \cdot \left[\frac{\partial^2 w^K}{\partial r^2} + \frac{1}{r}\frac{\partial w^K}{\partial r}\right] = -D\nabla^2 w^K \tag{12}$$

By substituting Eq. (11) into Eq. (3), the governing differential equation for the Kirchhoff plate is given by (Timoshenko and Woinowsky-Krieger, 1959):

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 w^K}{\partial r^2} + \frac{1}{r}\frac{\partial w^K}{\partial r}\right) = \frac{q}{D} \tag{13}$$

To consist of:

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w^K}{\partial r}\right)\right]\right\} = \frac{q}{D}$$

Thin plate bending solutions can be obtained by solving this governing equation with the boundary conditions. The typical boundary conditions for the axisymmetric bending of annular plates are listed in Table 1.

Reissner Plate Theory

Reissner developed a first order shear deformable plate model to analyze the effects of increasing thickness. In this model, the Kirchhoff assumption is relaxed to allow for the rotation of normal to the mid-plane during deformation. Reissner plate theory accounts for the shear deformation in the thickness in the simplest way.

The equilibrium equations for the plate bending, based on the Reissner plate theory, are given by Eqs. (3) and (4). In the Reissner plate theory, the stress resultant–displacement relationships for homogeneous plates are given as follows:

$$M_r^R = D \cdot \left[\frac{\partial \varphi_r^R}{\partial r} + \nu \frac{\varphi_r^R}{r}\right] + \frac{\nu h^2}{10(1-\nu)} q \tag{14}$$

$$M_\theta^R = D \cdot \left[\frac{\varphi_r^R}{r} + \nu \frac{\partial \varphi_r^R}{\partial r}\right] + \frac{\nu h^2}{10(1-\nu)} q \tag{15}$$

where ϕ_r^R: is the equivalent change of the slopes of the normal about the mid-surface. q: is the transverse load and the superscript R: shows the Reissner plate quantities.

By substituting Eqs. (14) and (15) into Eq. (4), one may get an alternative expression for the shear force:

$$Q_r^R = D \cdot \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\varphi_r^R\right)\right] + \frac{\nu h^2}{10(1-\nu)}\frac{\partial q}{\partial r} \tag{16}$$

Based on the first order shear deformable plate theory, the equivalent change of the slopes of the normal about the mid-surface is given as follows (Timoshenko and Woinowsky-Krieger, 1959):

$$\varphi_r^R = -\frac{\partial w^R}{\partial r} + \frac{12}{5}\frac{1+\nu}{\mathrm{Eh}} Q_r^R \tag{17}$$

where w^R: is the average transverse displacement. By substituting Eq. (17) into Eqs. (14) and (15), the bending moment to displacement relations can be obtained as:

Table 1. Typical boundary conditions for annular plates

	Clamped Boundary	Simply Supported Boundary	Free Boundary
$r = a$	$W = 0$ $dw/dr = 0$	$W = 0$ $M_r = M_a$	$Q_r = Q_a$ $M_r = M_a$
$r = b$	$W = 0$ $dw/dr = 0$	$W = 0$ $M_r = M_b$	$Q_r = Q_b$ $M_r = M_b$

$$M_r^R = -D \cdot \left[\frac{\partial^2 w^R}{\partial r^2} + \frac{\nu}{r} \frac{\partial w^R}{\partial r} \right] + \frac{h^2}{5} \frac{\partial Q_r^R}{\partial r} - \frac{\nu h^2}{10(1-\nu)} q \tag{18}$$

$$M_\theta^R = -D \cdot \left[\nu \frac{\partial^2 w^R}{\partial r^2} + \frac{1}{r} \frac{\partial w^R}{\partial r} \right] + \frac{h^2}{5} \frac{Q_r^R}{r} - \frac{\nu h^2}{10(1-\nu)} q \tag{19}$$

By substituting Eqs. (18) and (19) into Eq. (4), the shear force–displacement relation can be obtained as:

$$Q_r^R = -D \frac{\partial}{\partial r} \left[\nabla^2 w^R \right] - \frac{2-\nu}{1-\nu} \frac{h^2}{10} \frac{\partial q}{\partial r} \tag{20}$$

The Marcus moment M^R : is defined as:

$$\mathsf{M}^R = \frac{M_r^R + M_\theta^R}{1+\nu} = D \cdot \left[\frac{\partial \varphi_r^R}{\partial r} + \frac{\varphi_r^R}{r} \right] + \frac{\nu h^2}{5(1-\nu^2)} q = -D \nabla^2 w^R - \frac{h^2}{5(1-\nu^2)} q \tag{21}$$

Eqs. (3) and (20) can be combined to provide the fourth-order governing equation for the Reissner plate deflection:

$$D \nabla^4 w^{\mathrm{R}} = q - \frac{h^2}{10} \frac{2-\nu}{1-\nu} \nabla^2 q \tag{22}$$

to consist of:

$$\nabla^2 \left[\mathsf{M}^R - \frac{\nu h^2}{10(1+\nu)} q \right] = -q \tag{23}$$

Finally, by combining Eqs. (17) and (3), one may write (Reismann, 1988):

$$\frac{5}{12} \frac{\mathrm{Eh}}{1+\nu} \frac{\partial}{\partial r} \left[r \varphi_r^R + r \frac{\partial w^R}{\partial r} \right] = -\mathrm{r.q} \tag{24}$$

By introducing two generalized displacement functions F and f, the sixth-order governing equation can be split into two partial differential equations. One is the plate bending equation with F only; the other is the membrane on elastic foundation equation with f only. One can easily obtain the homogeneous solutions for these equations:

$$D \nabla^4 F = q \tag{25}$$

$$\nabla^2 f - \frac{10}{h^2} f = 0 \tag{26}$$

By inserting Eqs. (25) and (26) into Eqs. (17) and (22), values of average transverse displacement and the equivalent change of the slopes of the normal on the mid-surface are equal to:

$$\varphi_r^R = \frac{\partial (f)}{\partial r} - \frac{10}{h^2} \frac{\int rf dr}{r} - \frac{\partial (F)}{\partial r} - \frac{h^2}{10} \frac{\nu}{(1-\nu)} \frac{\partial \left(\nabla^2 F \right)}{\partial r} \tag{27}$$

$$w^R = F - \frac{h^2}{10} \frac{2-\nu}{1-\nu} \cdot \nabla^2 F \tag{28}$$

Eventually, the generalized stress-displacement relations can be expressed by *F* and *f*.

COMPLETE SOLUTIONS AND COMPLETE SETS

The T-complete set is derived by solving the homogeneous equations of a problem. This can be done using the method of separation of variables (Finlayson, 1972). The fundamental solutions with respect to variable θ are sin(nθ), cos(nθ), with $n = 0, 1, 2, ...,$ being the separation parameter. Solution functions for variable r are obtained from the fourth-order differential equation for the Kirchhoff plate or a set of fourth and second-order equations for the Reissner plate. For each separation parameter n, it deals with ordinary differential equations so the demanded number of fundamental solutions is equal to the order of the equation. The solution of the problem of the bending of an annular plate is a linear combination of the fundamental solutions. On the other hand, some of the functions may be eliminated due to physical reasons, e.g., a trial function tends to infinity at a certain point, whereas a finite result is expected (Wroblewski, 2005).

The T-complete set of solutions for the ring domain problem is as follows:

$$\begin{aligned} \mathrm{B_F} = [\; & 1, r^2, Lnr, r^2 Lnr \;;\; \mathrm{r}Cos\theta, \mathrm{r}Sin\theta, \\ & r^{-1}Cos\theta, r^{-1}Sin\theta, r^3 Cos\theta, r^3 Sin\theta, \\ & rLnrCos\theta, rLnrSin\theta \;;\; r^n Cos(n\theta), \\ & r^n Sin(n\theta), r^{-n}Cos(n\theta), r^{-n}Sin(n\theta), \\ & r^{n+2}Cos(n\theta), r^{n+2}Sin(n\theta), \\ & r^{-n+2}Cos(n\theta), r^{-n+2}Sin(n\theta) \;] \end{aligned} \tag{29}$$

where n: indices vary in the range $\{2,3,\ldots\}$.

INDIRECT TREFFTZ FORMULATION

The original formulation presented by Trefftz in 1926 is considered to be the indirect Trefftz method. The solution to the problem is estimated by the superposition of the functions satisfying the governing equation. Thus, the unknown parameters are determined so that the approximate solution satisfies the boundary conditions by the collocation, the least square or the Galerkin method (Kita and Kamiya, 1995). To illustrate the weighted residual procedure (Banerjee, 1981), we should consider the determination of a function (u), which may be a quantity within a region Ω bounded, by Γ, defined by the general equation:

$$L(u) = 0 \qquad \text{in } \Omega \tag{30}$$

This is subject to the boundary conditions:

$$\begin{aligned} w &= \bar{w} && \text{on } \Gamma_{\mathrm{w}} \\ \varphi_{\mathrm{s}} &= \bar{\varphi}_{\mathrm{s}} && \text{on } \Gamma_{\varphi\mathrm{s}} \\ \varphi_{\mathrm{n}} &= \bar{\varphi}_{\mathrm{n}} && \text{on } \Gamma_{\varphi\mathrm{n}} \\ M_n &= \bar{M}_n && \text{on } \Gamma_{Mn} \\ M_{ns} &= \bar{M}_{ns} && \text{on } \Gamma_{Mns} \\ Q_n &= \bar{Q}_n && \text{on } \Gamma_{Qn} \end{aligned} \tag{31}$$

These conditions are obtained from the boundary conditions mentioned before. The operator L may be either a differential or integral operator and is either linear or nonlinear in nature. If u^0 is to some degree of approximation, Eqs. (30) and (31) will not be satisfied exactly. To determine the approximate solution of u^0, some weighted integral of errors is set to zero, so that:

$$\begin{aligned} & \int_{\Omega} \mathrm{w}_1^* \left(L(u^0)\right) \cdot d\Omega + \int_{\Gamma_{\mathrm{w}}} \mathrm{w}_2^* \left(\mathrm{w}^0 - \bar{\mathrm{w}}\right) \cdot d\Gamma + \\ & \int_{\Gamma_{\varphi s}} \mathrm{w}_3^* \left(\varphi_s^0 - \bar{\varphi}_s\right) \cdot d\Gamma + \int_{\Gamma_{\varphi n}} \mathrm{w}_4^* \left(\varphi_n^0 - \bar{\varphi}_n\right) \cdot d\Gamma + \\ & \int_{\Gamma_{Mn}} \mathrm{w}_5^* \left(M_n^0 - \bar{M}_n\right) d\Gamma + \int_{\Gamma_{Mns}} \mathrm{w}_6^* \left(M_{ns}^0 - \bar{M}_{ns}\right) \cdot d\Gamma \\ & + \int_{\Gamma_{Qn}} \mathrm{w}_7^* \left(Q_n^0 - \bar{Q}_n\right) \cdot d\Gamma = 0 \end{aligned} \tag{32}$$

where $\mathrm{w}_{\mathrm{i}}^*$ (i = 1-7): are a set of independent weighting functions and in this case, the operator L: is defined as follows:

$$L(u) = D\nabla^4 u - q$$

The approximate solution of the generalized displacements F can be expressed by a series as:

$$F = \sum N_{Fi} a_i \tag{33}$$

where a_i: is undetermined coefficient and N_F: is the complete set of Trefftz functions. For any function N_F, then

$$\nabla^4 N_{Fi} = 0$$

By substituting Eq. (33) into deflection and stress–displacement equations and transforming the polar coordinate system to the local coordinate system (*n,s*), the variables in Eq. (32) can be written as:

$$\begin{aligned} &w^0 = \sum N_{1i} a_i \quad \varphi_s^{\,0} = \sum N_{2i} a_i \\ &\varphi_n^{\,0} = \sum N_{3i} a_i \;\; M_n^{\,0} = \sum N_{4i} a_i \\ &M_{ns}^{\,0} = \sum N_{5i} a_i \;\; Q_n^{\,0} = \sum N_{6i} a_i \end{aligned} \tag{34}$$

where a: is the undetermined coefficient and functions of N_{ij} (i = 1-6 , j =1,2,3,..) can be obtained by putting the T-complete functions into deflection and stress-displacement equations. In the Galerkin method, the Trefftz functions are also used as the weighting functions such that (Ghannadiasl and Noorzad, 2009):

$$\begin{aligned} &\mathrm{w}_1^* = \mathrm{N}_{1j} \quad , \; \mathrm{w}_2^* = \mathrm{N}_{6j} \quad , \; \mathrm{w}_3^* = -\mathrm{N}_{5j} \\ &\mathrm{w}_4^* = -\mathrm{N}_{4j} \;\; , \; \mathrm{w}_5^* = \mathrm{N}_{3j} \quad , \; \mathrm{w}_6^* = \mathrm{N}_{2j} \\ &\mathrm{w}_7^* = -\mathrm{N}_{1j} \end{aligned} \tag{35}$$

By substituting Eqs. (34) and (35) into Eq. (32), the matrix equation for the solution of the problem gets:

$$Ka = f \tag{36}$$

where

$$\begin{aligned} \boldsymbol{K} = &\int_{\Gamma_w} \mathrm{N}_{6j}\mathrm{N}_{1i} \cdot \mathrm{d}\Gamma - \int_{\Gamma_{\phi s}} \mathrm{N}_{5j}\mathrm{N}_{2i} \cdot \mathrm{d}\Gamma - \\ &\int_{\Gamma_{\phi n}} \mathrm{N}_{4j}\mathrm{N}_{3i} \cdot \mathrm{d}\Gamma + \int_{\Gamma_{Mn}} \mathrm{N}_{3j}\mathrm{N}_{4i} \cdot \mathrm{d}\Gamma + \\ &\int_{\Gamma_{Mns}} \mathrm{N}_{2j}\mathrm{N}_{5i} \cdot \mathrm{d}\Gamma - \int_{\Gamma_{Qn}} \mathrm{N}_{1j}\mathrm{N}_{6i} \cdot \mathrm{d}\Gamma \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{f} = &-\int_{\Omega} \mathrm{N}_1(q) \cdot \mathrm{d}\Omega + \\ &\int_{\Gamma_w} \mathrm{N}_{6j}\overline{\mathrm{w}} \cdot \mathrm{d}\Gamma - \int_{\Gamma_{\phi s}} \mathrm{N}_{5j}\bar{\phi}_s \cdot \mathrm{d}\Gamma - \\ &\int_{\Gamma_{\phi n}} \mathrm{N}_{4j}\bar{\phi}_n \cdot \mathrm{d}\Gamma + \int_{\Gamma_{Mn}} \mathrm{N}_{3j}\overline{M}_n \cdot \mathrm{d}\Gamma + \\ &\int_{\Gamma_{Mns}} \mathrm{N}_{2j}\overline{M}_{ns} \cdot \mathrm{d}\Gamma - \int_{\Gamma_{Qn}} \mathrm{N}_{1j}\overline{Q}_n \cdot \mathrm{d}\Gamma \end{aligned}$$

NUMERICAL EXAMPLES

In this section, the results of several simple examples are presented to validate the proposed method. The solutions were obtained using the Trefftz Galerkin Method (TGM). The boundary conditions include constant, linear and quadratic interpolations. The concentrated or distributed loads are either transformed into equivalent boundary integrals, by introducing a particular solution, by using a double node, or by placing non-conforming elements at a singular corner point (Jin and Cheung, 1999). For simplicity, ν = 0.3 and D = 1 have been used in the following examples.

Example 1: Simply supported annular plate under uniformly distributed load

A simply supported annular plate is subjected to a unit, uniformly distributed load. As a result of the presence of a smooth boundary, continuous elements have been adopted. The annular plate is supposed with the following characteristics:

$a = 1 \qquad D = 1$

$b = 0.1a \qquad h = 0.02a$

$q = 1 \qquad number\ of\ \ elements = 16$

The deflection, bending moment and rotation along the diameter of the simply supported annular plate subjected to a unit uniformly distributed load are shown in Figures 3-5 and Table 2. The comparison with exact results shows that the solutions are in good agreement. It is evaluated that the results are fairly close and the maximum difference is 0.1%. The exact solution based on the Kirchhoff plate theory is given by (Reddy, 2001):

$$w = \frac{q \cdot a^4}{64D}\left[-1 + \left(\frac{r}{a}\right)^4 + \frac{2\alpha_1}{1+\nu}\left(1 - \left(\frac{r}{a}\right)^2\right) - \frac{2\alpha_2\beta^2}{1-\nu}\log\left(\frac{r}{a}\right)\right]$$

$$M_r = -\frac{q \cdot a^2}{16}\left[(3+\nu)\cdot\left[1-\left(\frac{r}{a}\right)^2\right] - \beta^2(3+\nu)\left[1+\left(\frac{r}{a}\right)^2\right] + 4(1+\nu)\beta^2\kappa\left[1-\left(\frac{r}{a}\right)^2\right] + 4(1+\nu)\cdot\beta^2\log\left(\frac{r}{a}\right)\right]$$

$$\alpha_1 = (3+\nu)\cdot(1-\beta^2) - 4(1+\nu)\cdot\beta^2\kappa$$

$$\alpha_2 = (3+\nu) + 4(1+\nu)\cdot\kappa$$

$$\kappa = \frac{\beta^2}{1-\beta^2}\log(\beta) \qquad \beta = \frac{b}{a}$$

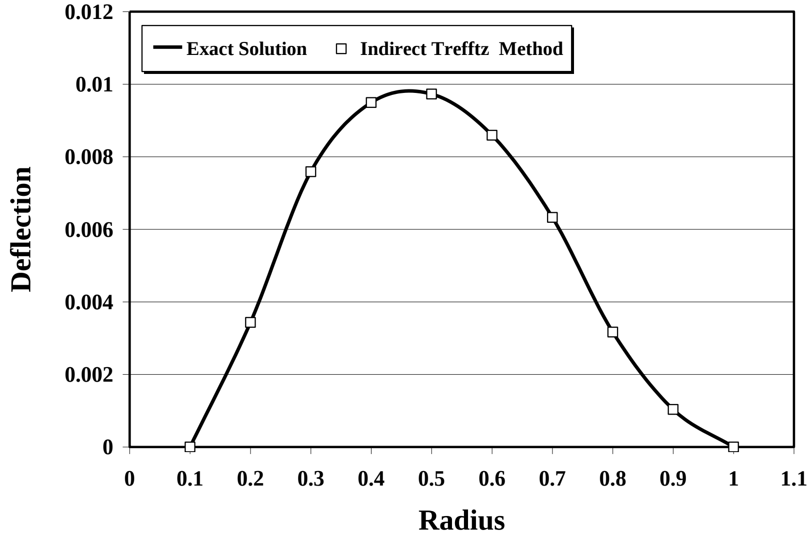

Fig. 3. Deflection along radius for simply supported annular plate ($b = 0.1a$, $h = 0.02a$)

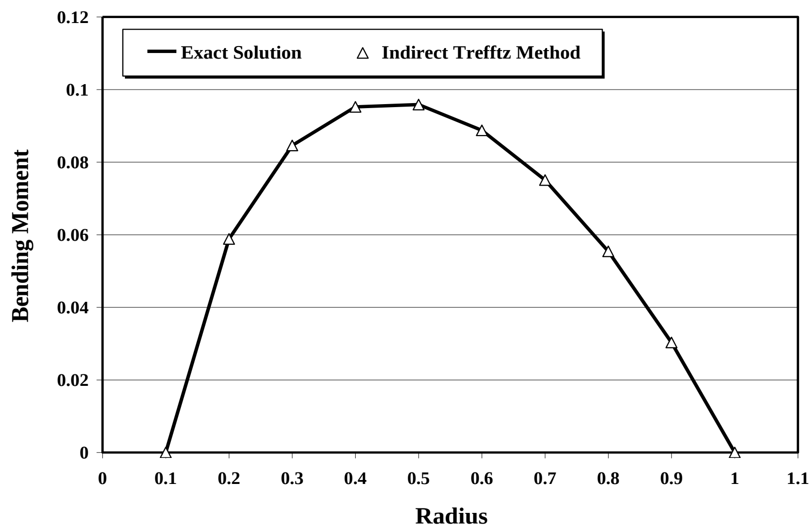

Fig. 4. Bending moment along radius for simply supported annular plate ($b = 0.1a$, $h = 0.02a$)

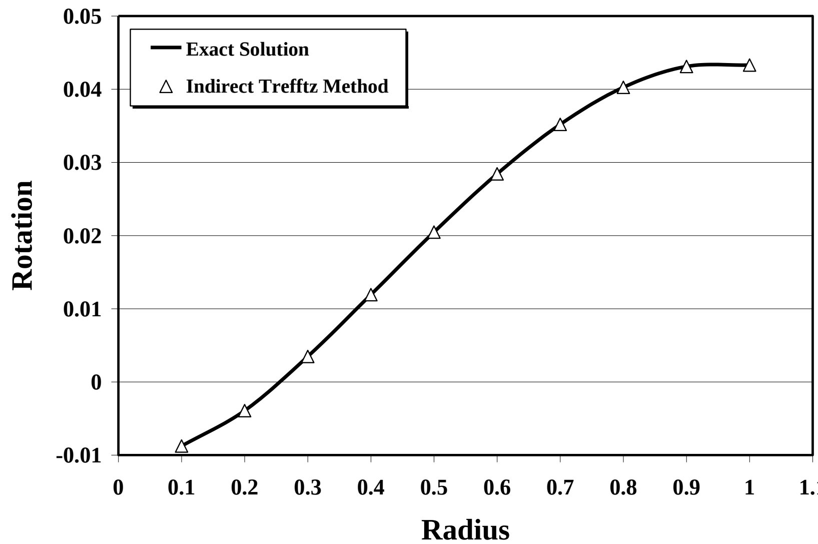

Fig. 5. Rotation along radius for simply supported annular plate (b = 0.1a, h = 0.02a)

Table 2. Simply supported annular plate under uniformly distributed load (a = 1, b = 0.1a, h = 0.02a)

		***r* = 0.2**	**0.4**	**0.5**	**0.7**	**0.9**	ε_{Max}
Indirect Trefftz Method	Deflection along radius	0.003434	0.009495	0.009729	0.006328	0.001036	0.0966 percent
	Bending moment along radius	0.058830	0.095220	0.095870	0.075054	0.030313	0.00989 percent
	Rotation along radius	-0.003930	0.011924	0.020474	0.035198	0.043105	0.02544 percent
Exact Solution	Deflection along radius	0.003433	0.009495	0.009728	0.006328	0.001035	---
	Bending moment along radius	0.058828	0.095221	0.095869	0.075054	0.030310	---
	Rotation along radius	-0.003931	0.011924	0.020474	0.035199	0.043107	---

Example 2: Annular plate under uniformly distributed load with different thicknesses

A simply supported annular plate with different thicknesses subjected to a uniformly distributed load is taken as an example. The corresponding discretization and the number of elements can be similarly adopted as in Example 1. The annular plate characteristics are as follows:

$$D = 1 \qquad b = 0.1a$$

$$q = 1 \qquad number\ of\ \ elements = 16$$

The deflection at several points of the plate, with thickness ratio *h/a* ranging from 0.0005 to 0.2, is given in Figure 6 and Table 3. It can be seen that the present method gives satisfactory results and that the method performs equally well for moderately thick plates as well as for very thin plates.

Table 3. Deflection at several points of the plate with thickness ratio h/a ranging from 0.0005 to 0.2

h/a	***W***				
	***r/a* = 0.2**	***r/a* = 0.4**	***r/a* = 0.5**	***r/a* = 0.7**	***r/a* = 0.9**
0.2	0.003434	0.009495	0.009729	0.006328	0.001636
0.1	0.006475	0.012164	0.012383	0.009180	0.002660
0.05	0.007234	0.012829	0.013045	0.009892	0.003483
0.01	0.007477	0.013042	0.013257	0.010119	0.003746
0.005	0.007484	0.013048	0.013264	0.010126	0.003754
0.002	0.007487	0.01305	0.013266	0.010128	0.003756
0.0005	0.007487	0.01305	0.013266	0.010129	0.003756
Kirchhoff Solution	0.007487	0.01305	0.013266	0.010129	0.003756

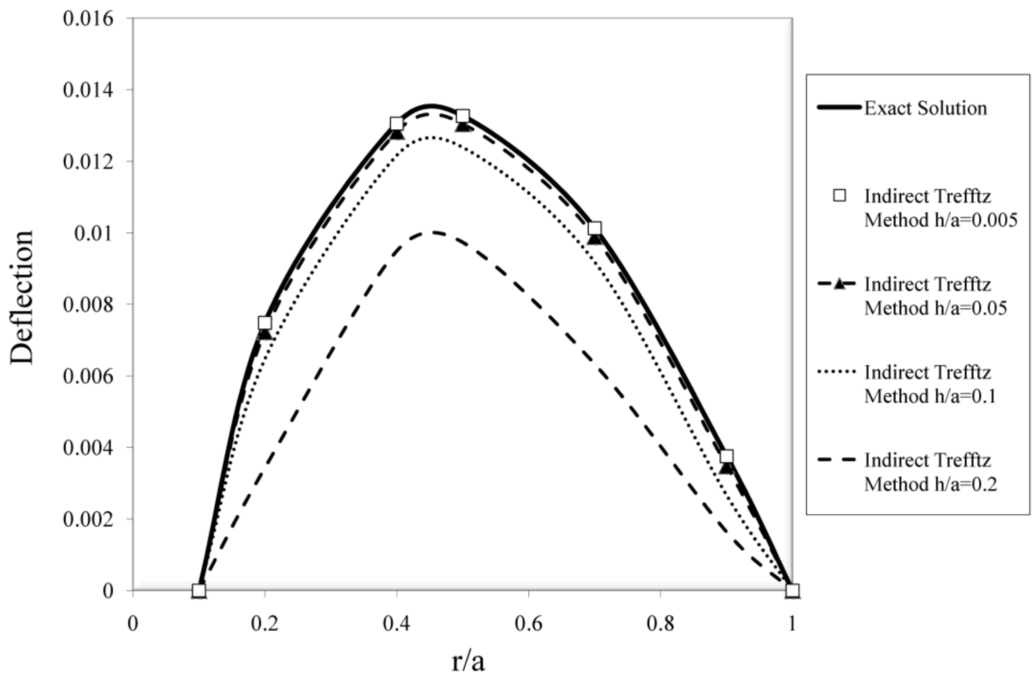

Fig. 6. Deflection of the annular plate with thickness ratio *h/a* ranging from 0.0005 to 0.2

Example 3. Annular plate with different ratio inner radius to outer radius

A simply supported annular plate with a different ratio of inner radius to outer radius, subjected to a uniformly distributed load is taken. The annular plate is supposed with the following characteristics:

$D = 1 \qquad h = 0.2a$

$q = 1 \qquad number\ of\ \ elements = 16$

The results of the deflection and bending moment at several points of the plate with ratios of inner radius to outer radius ranging from 0.05 to 0.2 are given in Tables 4 and 5.

It was evaluated that the indirect Trefftz boundary method gave satisfactory results. Furthermore, the method is accurate for holes that have a diameter smaller than the thickness of the plate.

Table 4. Deflection at several points of the plate with ratio b/a ranging from 0.05 to 0.2

b/a	***W***				
	***r/a* = 0.2**	***r/a* = 0.4**	***r/a* = 0.5**	***r/a* = 0.7**	***r/a* = 0.9**
0.05	0.013434	0.018132	0.017280	0.011313	0.001553
0.1	0.003434	0.009495	0.009729	0.006328	0.001636
0.15	0.001243	0.003978	0.004781	0.002987	0.001068
0.2	0.000000	0.001852	0.002452	0.001631	0.000808

Table 5. Bending Moment at Several Points of the Plate with Ratio b/a Ranging from 0.05 to 0.2.

b/a	***W***				
	***r/a* = 0.2**	***r/a* = 0.4**	***r/a* = 0.5**	***r/a* = 0.7**	***r/a* = 0.9**
0.05	0.0987832	0.1181801	0.1132742	0.084034	0.0329671
0.1	0.0588286	0.0952209	0.0958695	0.0750546	0.0303103
0.15	0.0278759	0.0772733	0.0822362	0.0680033	0.0282209
0.2	0.0000000	0.0609646	0.0698232	0.0615674	0.0263110

CONCLUSIONS

This paper presents the bending analysis of annular plates using the indirect Trefftz boundary method. The indirect approximation method is applied using the displacement functions F and f for an annular plate, and by adopting the Trefftz function and the T-complete set. The main conclusions of this paper can be summarized as follows:

1. Numerical solutions show that the present method is not only effective but also provides accurate numerical results.
2. The method is accurate for holes that have a diameter smaller than the thickness of the plate.

As a result of the use of the complete sets of solutions used as weighting and/or trial functions, there is no need to use the singular integral. Therefore, it is expected that this method can be a suitable method for the analysis of arbitrarily shaped plates.

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