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Rocking Rotation of a Rigid Disk Embedded in a Transversely Isotropic Half-Space

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ABSTRACT: The asymmetric problem of rocking rotation of a circular rigid disk embedded in a finite depth of a transversely isotropic half-space is analytically addressed. The rigid disk is assumed to be in frictionless contact with the elastic half-space. By virtue of appropriate Green's functions, the mixed boundary value problem is written as a dual integral equation. Employing further mathematical techniques, the integral equation is reduced to a well-known Fredholm integral equation of the second kind. The results related to the contact stress distribution across the disk region and the equivalent rocking stiffness of the system are expressed in terms of the solution of the obtained Fredholm integral equation. When the rigid disk is located on the surface or at the remote boundary, the exact closed-form solutions are presented. For verification purposes, the limiting case of an isotropic half-space is considered and the results are verified with those available in the literature. The jump behavior in the results at the edge of the rigid disk for the case of an infinitesimal embedment is highlighted analytically for the first time. Selected numerical results are depicted for the contact stress distribution across the disk region, rocking stiffness of the system, normal stress, and displacement components along the radial axis. Moreover, effects of anisotropy on the rocking stiffness factor are discussed in detail.

Keywords: Fredholm Integral Equation, Rigid Disk, Rocking Stiffness, Soil-Structure-Interaction, Transversely Isotropic.

INTRODUCTION

The soil-foundation interaction problem is an attractive subject for many researchers in the fields of civil engineering, geotechnical engineering, seismology, and applied mechanics. The results of such studies are of practical importance for the design of various structural elements like piles, anchors, foundations, and etc. under static and dynamic loading conditions. Moreover, determination of the equivalent stiffness of the system can significantly simplify further numerical simulations of soil-structure interaction.

For analytical treatment of soil-structure interaction problems, the foundation is usually modeled using a rigid or flexible

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disk. Generally, a rigid disk in contact with an elastic medium can be subjected to four different types of load or deformation; (i) normal translation, (ii) lateral translation, (iii) rocking rotation, and (iv) torsional rotation. Furthermore, the contact between the disk and medium is assumed to be either smooth (frictionless) (Selvadurai, 2009; Eskandari-Ghadi et al., 2010), or adhesive (perfectly bonded) (Fabrikant, 1997; Selvadurai, 1993) in the literature. For the case of infinite embedment of the rigid disk in an elastic solid (Selvadurai, 1980a), the results corresponding to both contact models are identical. However, for the adhesive contact assumption, when the disk rests on the surface (Gladwell, 1969) or embedded in a finite depth of elastic half-space (Selvadurai et al., 1991), the lateral translation and rocking rotation are coupled together. In other words, the lateral deformation of the rigid disk, which is formed by a concentrated lateral load, accompanies some rocking rotation, and vice versa. However, in the case of the smooth contact between the rigid disk and the elastic half-space, four aforementioned loading types are decoupled. In this context, the interaction problems of rigid and flexible disks with isotropic media have been thoroughly studied in the literature (e.g. Pak and Gobert, 1990; Pak and Saphores, 1991). The reader is referred to the extensive list of references cited in Pak and Gobert (1990).

The formation process of most kinds of rocks and soils under various environmental conditions like high multi-directional pressures, heat, cracking, and etc. results in their anisotropic behavior. Hence, geotechnical engineers are more interested in realistic analysis of these materials by considering anisotropic models. Moreover, the application of anisotropic materials in advanced technologies necessitates the study of their responses under static and dynamic loads. In practical applications, anisotropic materials are often modeled as transversely isotropic or orthotropic media.

Problems concerning a disk embedded in a transversely isotropic full-space under four possible types of loading are presented by Flavin and Gallagher (1976), Selvadurai (1980a,b; 1982; 1984) and Eskandari-Ghadi et al. (2011). It seems that apart from the coupled lateral translation and rocking rotation of a rigid disk in the adhesive contact with the surface of a transversely isotropic half-space, other cases related to the disk-half-space surface interaction have been studied. Rahimian et al. (2006) considered the torsion of a rigid disk rested on the surface of a transversely isotropic half-space. Recently, Katebi et al. (2010) and Eskandari-Ghadi and Ardeshir-Behrestaghi (2010) solved the axisymmetric interaction problem of a rigid circular disk embedded in a finite depth of a transversely isotropic half-space under normal static and time-harmonic loadings, respectively. To date, researches on the interaction of a rigid disk embedded in a finite depth of a transversely isotropic half-space are limited to simple axisymmetric loadings. Such problems related to the finite embedment of the rigid disk in a transversely isotropic half-space is an appropriate model to investigate the behavior of anchorage systems subjected to various types of loading.

In this paper, the analytical treatment of the asymmetric interaction of a rigid circular disk embedded in a finite depth of a transversely isotropic elastic half-space is addressed. The contact between the disk and the half-space is assumed to be smooth, and the rigid disk is subjected to a rocking moment. With the aid of appropriate Green's functions, the governing equations of the mixed boundary value problem are written as a dual integral equation. Employing Nobel's method (Nobel, 1963) and further mathematical techniques, the solution of the problem is given in terms of a well-known

Fredholm integral equation. The available results in the literature corresponding to the infinite embedment, surface contact, and isotropic half-space are exactly recovered as the limiting cases of the current study. Some numerical results for the rocking stiffness, contact stress distribution, and displacement fields are depicted. The effects of anisotropy on the rocking stiffness are also highlighted.

PROBLEM STATEMENT

Let's consider the relaxed treatment of a massless rigid disk of vanishingly small thickness and radius a embedded in a finite depth $s > 0$ of a homogeneous transversely isotropic linear elastic half-space under rocking rotation, see Figure 1. The rigid disk is located in the isotropy plane of the medium. The origin of the cylindrical coordinate system $O(r, \theta, z)$ is set on the surface in such a way that the z-axis points into the half-space and coincides with the axis of symmetry of the medium. The system is disturbed by a rigid-body rotation about the axis parallel to the medium Ω imposed on the disk due to the application of a rocking moment applied to the rigid disk. It is assumed that the contact between the rigid disk and the medium is smooth. Therefore, the rigid disk is under pure rotation about the y-axis without any lateral translations.

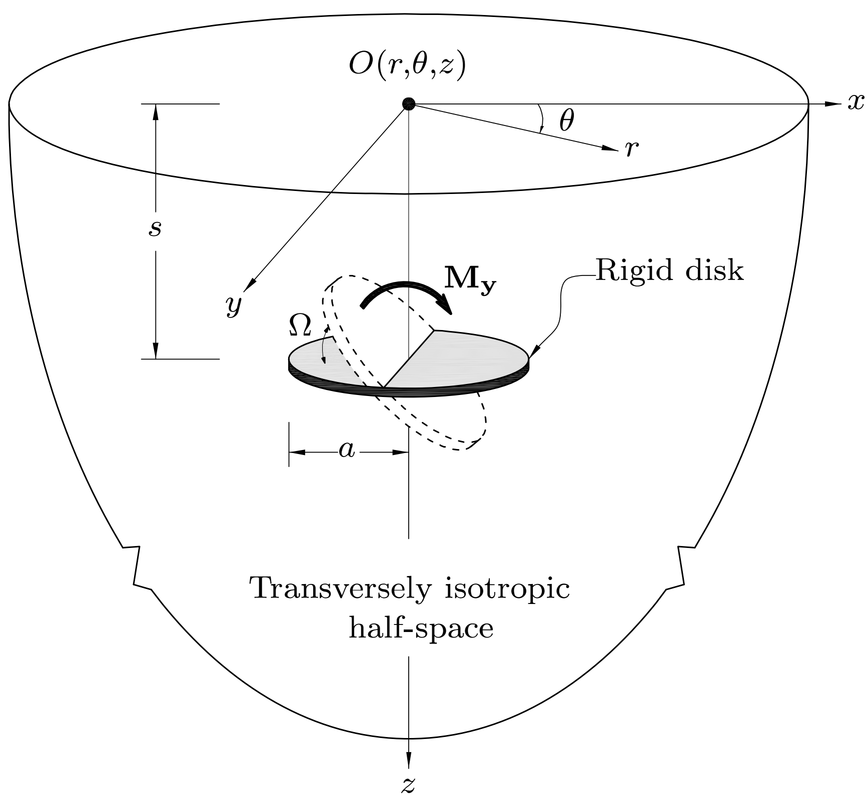


Fig. 1. Rocking rotation of a rigid disk embedded in a finite depth of a transversely isotropic half-space.

The mixed boundary conditions of the problem under consideration in terms of the displacement vector $\boldsymbol{u}$ and the Cauchy stress tensor $\boldsymbol{\sigma}$ can be written as:

$$u_z(r,\theta,s) = \Omega r\cos\theta\,, \qquad r<a, \tag{1}$$

For the rigid-body rotation of the disk;

$$u_z(r,\theta,s^-) = u_z(r,\theta,s^+)\,, \quad r\geq 0, \tag{2}$$

$$u_r(r,\theta,s^-) = u_r(r,\theta,s^+)\,, \quad r\geq 0, \tag{3}$$

$$u_\theta(r,\theta,s^-) = u_\theta(r,\theta,s^+)\,, \quad r\geq 0, \tag{4}$$

For the continuity of displacement components across the plane $z=s$; and:

$$\sigma_{zr}(r,\theta,s^-) = \sigma_{zr}(r,\theta,s^+)\,, \quad r\geq 0, \tag{5}$$

$$\sigma_{z\theta}(r,\theta,s^-) = \sigma_{z\theta}(r,\theta,s^+)\,, \quad r\geq 0, \tag{6}$$

$$\sigma_{zz}(r,\theta,s^-) - \sigma_{zz}(r,\theta,s^+) = R(r,\theta;s), \quad r<a, \tag{7}$$

For the continuity and jump conditions of stress components across the disk plane $z=s$. Here, the unknown $R(r,\theta;s)$ is the normal component of traction acting on the disk region. Moreover, the traction free condition on the surface ($z=0,\ r\geq 0$) implies that:

$$\sigma_{zz}(r,\theta) = \sigma_{zr}(r,\theta) = \sigma_{z\theta}(r,\theta) = 0, \tag{8}$$

where $0\leq\theta\leq 2\pi$. One must also satisfy the regularity conditions at infinity as:

$$\lim_{r,z\to\infty} u = 0, \qquad \lim_{r,z\to\infty} \sigma = 0\,, \tag{9}$$

Governing Integral Equation

All elastic fields of the problem can be expressed in the Fourier series expansion with respect to the angular coordinate θ, for instance:

$$R(r,\theta) = \sum_{m=-\infty}^{\infty} R_m(r)e^{im\theta}\,, \tag{10}$$

with similar expressions for the displacement and stress components. Moreover, considering the m-th order Hankel transform for sufficiently regular function $f(r)$ with respect to the radial coordinate:

$$\tilde{f}^m(\xi,z) = \int_0^\infty f(r,z)rJ_m(\xi r)\,dr, \tag{11}$$

where ξ is the transform parameter and J_m is the mth order Bessel function of the first kind, the following term is defined:

$$Z_m = \tilde{R}_m^m(\xi). \tag{12}$$

Now, according to the formulations used by Eskandari and Shodja (2010) as well as Katebi et al. (2010), the Fourier components of the displacement and stress fields for the buried lateral excitation in a finite depth $z=s$ can be written as:

$$u_{z_m} = \int_0^\infty \xi\Omega_2(z,\xi;s)\left(\frac{Z_m}{c_{44}}\right)J_m(r\xi)\,d\xi, \tag{13}$$

$$u_{r_m} - iu_{\theta_m} = \int_0^\infty \xi\gamma_3(z,\xi;s)\left(\frac{Z_m}{c_{44}}\right)J_{m-1}(r\xi)\,d\xi, \tag{14}$$

$$u_{r_m} + iu_{\theta_m} = -\int_0^\infty \xi\gamma_3(z,\xi;s)\left(\frac{Z_m}{c_{44}}\right)J_{m+1}(r\xi)\,d\xi \tag{15}$$

and

$$\sigma_{zz_m} = \int_0^\infty \xi\left[c_{33}\frac{d\Omega_2(z,\xi;s)}{dz} - c_{13}\xi\gamma_3(z,\xi;s)\right]\left(\frac{Z_m}{c_{44}}\right)J_m(r\xi)d\xi, \tag{16}$$

$$\sigma_{zr_m} - i\sigma_{z\theta_m} = \int_0^\infty \xi\left[\xi\Omega_2(z,\xi;s) + \frac{d\gamma_3(z,\xi;s)}{dz}\right]Z_mJ_{m-1}(r\xi)d\xi, \tag{17}$$

$$\sigma_{zr_m} + i\sigma_{z\theta_m} = -\int_0^\infty \xi\left[\xi\Omega_2(z,\xi;s) + \frac{d\gamma_3(z,\xi;s)}{dz}\right]Z_mJ_{m+1}(r\xi)d\xi, \tag{18}$$

where the kernel functions $\Omega_2(z,\xi;s)$ and $\gamma_3(z,\xi;s)$ are given by Eskandari and Shodja (2010) as:

$$\Omega_2(z,\xi;s) = \frac{1}{2\alpha_4(s_1^2-s_2^2)\xi}\Big\{s_1\varrho_2e^{-s_1\xi|z-s|} - s_2\varrho_1e^{-s_2\xi|z-s|} - \frac{s_1+s_2}{s_1-s_2}\left[s_1\varrho_2e^{-s_1\xi(z+s)} + s_2\varrho_1e^{-s_2\xi(z+s)}\right] + \frac{2s_1s_2}{s_1-s_2}\left[\varrho_2\frac{\alpha_3-\varrho_1}{\alpha_3-\varrho_2}e^{-\xi(s_1z+s_2s)} + \varrho_1\frac{\alpha_3-\varrho_2}{\alpha_3-\varrho_1}e^{-\xi(s_2z+s_1s)}\right]\Big\}, \tag{19}$$

$$\gamma_3(z,\xi;s) = \frac{\alpha_3}{2\alpha_4\xi(s_1^2-s_2^2)}\Big\{sgn(z-s)\left[e^{-s_1\xi|z-s|} - e^{-s_2\xi|z-s|}\right] - \frac{s_1+s_2}{s_1-s_2}\left[e^{-s_1\xi(z+s)} + e^{-s_2\xi(z+s)}\right] + \frac{2}{s_1-s_2}\left[s_2\frac{\alpha_3-\varrho_1}{\alpha_3-\varrho_2}e^{-\xi(s_1z+s_2s)} + s_2\frac{\alpha_3-\varrho_2}{\alpha_3-\varrho_1}e^{-\xi(s_2z+s_1s)}\right]\Big\}, \tag{20}$$

where $\varrho_i = 1-\alpha_4s_i^2, i = 1,2$ and $\alpha_1 = \frac{c_{11}}{c_{44}}$, $\alpha_2 = \frac{c_{66}}{c_{44}}$, $\alpha_3 = \frac{c_{13}+c_{44}}{c_{44}}$, $\alpha_4 = \frac{c_{33}}{c_{44}}$. Here, c_{11}, c_{12}, c_{13}, c_{33}, c_{44} and $c_{66} = (c_{11}-c_{12})/2$ are the elastic constants, $s_0 = \sqrt{\alpha_2}$, and s_1 and s_2 are the roots of the following equation with positive real parts:

$$\alpha_4s^4 + (\alpha_3^2 - \alpha_1\alpha_4 - 1)s^2 + \alpha_1 = 0. \tag{21}$$

In view of the positive-definiteness of the strain energy, s_1 and s_2 are neither zero nor pure imaginary numbers. It is worth noting that Green's functions obtained by Eskandari and Shodja (2010) and Katebi et al. (2010) are identical, however, the notations used here are taken directly from the former paper.

The boundary condition (1) implies that:

$$Z_1 = Z_{-1}, \qquad Z_m = 0, \qquad m \neq \pm 1, \tag{22}$$

and subsequently, the vertical component of the displacement field can be expressed as:

$$u_z(r,\theta,z) = 2u_{z_1}cos\theta. \tag{23}$$

It is evident that the Eqs. (13-18) satisfy all the boundary conditions except Eqs. (1) and (7). By virtue of Sonine's integral as in Nobel (1963), these two remained boundary conditions can be reduced to a dual integral equation:

$$\int_0^\infty \frac{1}{\sqrt{\xi}}[1 + H_s(\xi;s)]Z(\xi;s)J_{\frac{1}{2}}(r\xi)\,d\xi = \frac{2\alpha_4(s_1+s_2)}{s_1s_2\alpha_4+1}\Omega\sqrt{\frac{2r}{\pi}}, \quad (24)$$

For $r < a$; and

$$\int_0^\infty \frac{1}{\sqrt{\xi}}Z(\xi;s)J_{1/2}(r\xi)\,d\xi = 0, \quad (25)$$

For $r > a$; in which

$$H_s(\xi;s) = \frac{-1}{(s_1-s_2)(s_1\varrho_2-s_2\varrho_1)}\Big\{(s_1+s_2)\big(s_1\varrho_2e^{-2s_1\xi s}+s_2\varrho_1e^{-2s_2\xi s}\big) - 2s_1s_2\left(\varrho_1\frac{\alpha_3-\varrho_2}{\alpha_3-\varrho_1} + \varrho_2\frac{\alpha_3-\varrho_1}{\alpha_3-\varrho_2}\right)e^{-(s_1+s_2)\xi s}\Big\}, \quad (26)$$

and

$$Z(\xi;s) = \frac{\xi Z_1(\xi;s)}{c_{44}}. \quad (27)$$

It is worth mentioning that the function $H_s(\xi;s)$ is indeterminate for identical values of s_1 and s_2. Therefore, by taking the limit $s_1 \to s_2$the relations pertinent to the case $s_1 = s_2$ is given by:

$$H_s(\xi;s) = \frac{-e^{-2s_1\xi s}}{(1+\alpha_4s_1^2)(\alpha_3-\varrho_1)^2}\{2s^2s_1^2\varrho_1(\alpha_3-\varrho_1)^2\xi^2 - (1+\alpha_4s_1^2)[2ss_1(\alpha_3-\varrho_1)^2\xi-\alpha_3^2-\varrho_1^2] - \alpha_3(2+6\alpha_1\alpha_4)\}. \quad (28)$$

The solution of dual integral Eqs. (23) and (24) yields the transformed contact stress distributions $Z(\xi)$ acting on the disk. Subsequently, the unknown traction components $R(r,\theta;s)$ can be obtained by recourse to (26) and the inverse Hankel transform. It is clear that one can use this traction component in conjunction with Green's functions introduced by Eskandari and Shodja (2010) to easily determine all the elastic fields of the proposed problem.

Further Reduction of the Dual Integral Equation

Let’s define

$$\int_0^\infty \frac{1}{\sqrt{\xi}}Z(\xi;s)J_{1/2}(r\xi)\,d\xi = \begin{cases}\sqrt{\frac{2}{\pi r}}\eta_s(r), & r < a,\\ 0, & r > a,\end{cases} \quad (29)$$

where $\eta_s(r)$ is an unknown function. Employing the Hankel transform yields:

$$Z(\xi;s) = \sqrt{\frac{2\xi^3}{\pi}}\int_0^a \sqrt{r}\eta_s(r)J_{1/2}(\xi r)\,dr, \quad (30)$$

Substituting Eq. (31) into Eq. (25), the dual integral equation can be reduced to a Fredholm integral equation of the second kind as:

$$\eta_s(r) + \int_0^a K_s(r,\rho)\eta_s(\rho)\,d\rho = f_s(r), \tag{31}$$

in which $K_s(r,\rho)$ and $f_s(r)$ are:

$$K_s(r,\rho) = \sqrt{r\rho}\int_0^\infty \xi H_s(\xi) J_{1/2}(r\xi) J_{1/2}(\rho\xi)\,d\xi, \tag{32}$$

$$f_s(r) = \frac{2\alpha_4(s_1+s_2)}{s_1 s_2 \alpha_4 + 1}\Omega r. \tag{33}$$

The kernel function $K_s(r,\rho)$ is evaluated in closed form as:

$$K_s(r,\rho) = k(r-\rho) - k(r+\rho), \tag{34}$$

in which

$$k(x) = \frac{2s(s_1+s_2)}{\pi(s_1-s_2)(s_1\varrho_2 - s_2\varrho_1)}\left\{\left[\frac{\varrho_1(\alpha_3-\varrho_2)}{\alpha_3-\varrho_1} + \frac{\varrho_2(\alpha_3-\varrho_1)}{\alpha_3-\varrho_2}\right]\frac{s_1 s_2}{x^2 + s^2(s_1+s_2)^2} - \left[\frac{s_2^2\varrho_1}{x^2+4s^2s_2^2} + \frac{s_1^2\varrho_2}{x^2+4s^2s_1^2}\right]\right\}. \tag{35}$$

By means of Eq. (29) and the identity of:

$$\int_0^\infty \sin(\xi\rho) J_0(r\xi)\,d\xi = \begin{cases} 0, & \rho < r, \\ \dfrac{1}{\sqrt{\rho^2 - r^2}}, & \rho > r, \end{cases} \tag{36}$$

The contact stress distribution across the disk region becomes:

$$R(r,\theta;s) = -\frac{4c_{44}\cos\theta}{\pi}\frac{d}{dr}\int_r^a \frac{\eta_s(\rho)}{\sqrt{\rho^2 - r^2}}\,d\rho, \tag{37}$$

which can be equivalently rewritten as:

$$R(r,\theta;s) = \frac{4c_{44}\cos\theta}{\pi}\left\{\frac{a\eta_s(a)}{r\sqrt{a^2-r^2}} - \frac{1}{r}\int_r^a \frac{\rho\eta'(\rho)\,d\rho}{\sqrt{\rho^2-r^2}}\right\}. \tag{38}$$

The rocking stiffness of the system under consideration is another important element to be obtained. The rocking stiffness is defined as the ratio of the total applied moment about the y-axis to the sustained rotation. The resultant moment applied to the disk can be obtained by:

$$M_y = \int_0^a \int_0^{2\pi} R(r,\theta;s) r^2 \cos\theta\,d\theta\,dr. \tag{39}$$

By recourse to Eq. (40), the rocking moment is expressed as:

$$M_y = 8c_{44}\int_0^a r\eta_s(r)\,dr. \tag{40}$$

The dimensionless rocking stiffness (K_{RR}) is defined as:

$$K_{RR} = \frac{M_y}{c_{44}a^3\Omega} = \frac{8}{a^3\Omega}\int_0^a r\eta_s(r)\,dr. \tag{41}$$

Limiting Cases

Rigid disk embedded in a transversely isotropic full-space

By taking the limit $s \to \infty$, one can obtain the closed-form solution pertinent to the rocking rotation of a rigid disk embedded in a transversely isotropic full-space. Taking the limits, one can find:

$$\eta_\infty(r) = \frac{2\alpha_4(s_1 + s_2)}{\alpha_4 s_1 s_2 + 1}\Omega r. \tag{42}$$

By substituting Eq. (41) into Eqs. (38) and (41), the closed-form solutions of contact load distribution and the dimensionless rocking stiffness of the system are obtained:

$$R(r,\theta;\infty) = \frac{8\alpha_4 c_{44} cos\theta(s_1 + s_2)}{\alpha_4 s_1 s_2 + 1}\left(\frac{\Omega r}{\sqrt{a^2 - r^2}}\right), \tag{43}$$

and

$$K_{RR}^{\infty} = \frac{16\alpha_4(s_1 + s_2)}{3(\alpha_4 s_1 s_2 + 1)}. \tag{44}$$

These results are in same agreement with those presented by Selvadurai (1980a).

Rigid disk resting on the surface of a transversely isotropic half-space

Let's take the limit $s \to 0^+$ for the kernel function (36):

$$\lim_{s\to 0^+} k(x) = \frac{\delta(x)}{(1 + \alpha_4 s_1 s_2)\varrho^*}\{(1 + \alpha_4 s_1 s_2)[\alpha_3^2 + \varrho_1\varrho_2] - \alpha_3\big[\alpha_4 s_1 s_2\big(2 + \alpha_4(s_1^2 + s_2^2 + s_1 s_2)\big) + \varrho_1 + \varrho_2\big]\}, \tag{45}$$

where $\delta(x)$ denotes Dirac delta function and $\varrho^* = (\alpha_3 - \varrho_1)(\alpha_3 - \varrho_2)$. By substituting Eq. (45) into Eq. (30), the solution of the Fredholm integral equation becomes:

$$\eta^{0^+}(r) = \begin{cases} \dfrac{\varrho^*\Omega r}{\alpha_3\alpha_4 s_1 s_2(s_1+s_2)}, & r < a, \\ \dfrac{4\alpha_4\varrho^*(s_1+s_2)\Omega a}{\varrho^*(1+\alpha_4 s_1 s_2)+2\alpha_3\alpha_4^2 s_1 s_2(s_1+s_2)^2}, & r \to a, \end{cases} \tag{46}$$

which shows a finite jump at the edge of the rigid disk for the case of an infinitesimal embedment of the rigid disk. On the other hand, if one sets $s = 0$ and rearranges the related governing integral equation, the pertinent solution is given by:

$$\eta^0(r) = \frac{\varrho^*\Omega r}{\alpha_3\alpha_4 s_1 s_2(s_1 + s_2)}, \tag{47}$$

which is no longer discontinuous at the edge of the disk. Subsequently, one can find

$$\eta^0(r) = \frac{\varrho^*\Omega r}{\alpha_3\alpha_4 s_1 s_2(s_1 + s_2)}, \tag{48}$$

$$\lim_{s\to 0^+}\lim_{r\to a}\frac{R(r,\theta;s)}{R(r,\theta;0)} = \frac{4\alpha_3\alpha_4^2 s_1 s_2(s_1 + s_2)^2}{\varrho^*(1 + \alpha_4 s_1 s_2) + 2\alpha_3\alpha_4^2 s_1 s_2(s_1 + s_2)^2}. \tag{49}$$

Substitution of Eq. (47) into Eq. (41) yields the dimensionless rocking stiffness as:

$$K_{RR}^0 = \frac{8\varrho^*}{3\alpha_3\alpha_4 s_1 s_2(s_1 + s_2)}, \tag{50}$$

Rigid disk embedded in isotropic half-spaces

For isotropic materials with shear modulus μ, and Poisson's ratio ν, the following relations hold:

$$c_{11} = c_{33} = \frac{2\mu(1-\nu)}{1-2\nu}, \quad c_{12} = c_{13} = \frac{2\mu\nu}{1-2\nu}, \quad c_{44} = c_{66} = \mu, \tag{51}$$

which subsequently yields:

$$s_0 = s_1 = s_2 = 1. \tag{52}$$

Using these relations, the obtained Fredholm integral equation is adapted to the problem concerning isotropic materials considered by Pak and Saphores (1991). For instance, the kernel function (34) for isotropic materials is reduced to:

$$k(x) = \frac{2}{\pi}\left\{\frac{2}{3-4\nu}\left[\frac{8s^5 - 6s^3x^2}{(x^2+4s^2)^3}\right] + \frac{5-12\nu+8\nu^2}{3-4\nu}\left[\frac{s}{x^2+4s^2}\right] + \frac{4s^3 - sx^2}{(x^2+4s^2)}\right\}, \tag{53}$$

which is in exact agreement with the result presented by Pak and Saphores (1991).

NUMERICAL RESULTS AND DESCUSSION

To the best of the authors' knowledge, the obtained Fredholm integral Eq. (32) cannot be treated analytically for the case of general embedment depths. However, the obtained integral equation can be solved numerically by the conventional methods (Atkinson, 1997). Introducing the dimensionless parameters $\bar{r} = r/a$, and $\bar{s} = s/a$, and dividing the interval $\bar{r} \in [0,1)$ into N equal segments, the Fredholm integral equation is reduced to a system of linear algebraic equations as:

$$[A]\{\phi\} = \{f\}, \tag{54}$$

where

$$\phi_j = \frac{\eta_s(t_i)}{\Omega}, \qquad t_i = \frac{2i-1}{2N}, \quad i,j = 1,2,\dots,N,$$
$$A_{ij} = \delta_{ij} + \frac{1}{N}K_s\left(t_i, t_j; \bar{s}\right), \tag{55}$$
$$f_i = \frac{2\alpha_4(s_1+s_2)}{1+\alpha_4 s_1 s_2} t_j.$$

in which δ_{ij} is the Kronecker delta.

With the aid of the introduced numerical scheme, the solution of the problem can be obtained for any type of transversely isotropic materials. Table 1 lists the properties of some transversely isotropic materials (Ding et al., 2006).

Table 1. Properties of tested specimens.

Material	c_{11} (GPa)	c_{12} (GPa)	c_{13} (GPa)	c_{33} (GPa)	c_{44} (GPa)
Barium titanate	168	78	71	189	5.46
Beryl rock	41.3	14.7	10.1	36.2	10.0
Composite	13.6	7	5.47	144	6.01
Graphite/epoxy	13.92	6.92	6.44	160.7	7.07
$NbSe_2$	106	14	31	54	19.5

In order to illustrate the effects of material anisotropy, the dimensionless rocking stiffness versus the dimensionless depth parameter (s/a) is plotted in Figure 2. The gray zone in this figure is pertinent to the results corresponding to the isotropic materials. By looking at this figure it is evident that for some transversely isotropic materials neglecting the real anisotropic behavior may lead to unrealistic predictions. In other words, simplified isotropic models may lead to results completely different from the real behavior of transversely isotropic media.

The contact stress distribution $R(r,\theta)$ across the disk embedded in a Beryl rock half-space is depicted in Figure 3. The stress magnitude tends to infinity near the disk edge and it has a singular behavior of order $O(1/\sqrt{a-r})$.

The normal stress and displacement components along the radial axis at different depths of a Beryl rock half-space for the same depth $s/a=1$ are illustrated in Figures 4 and 5, respectively.

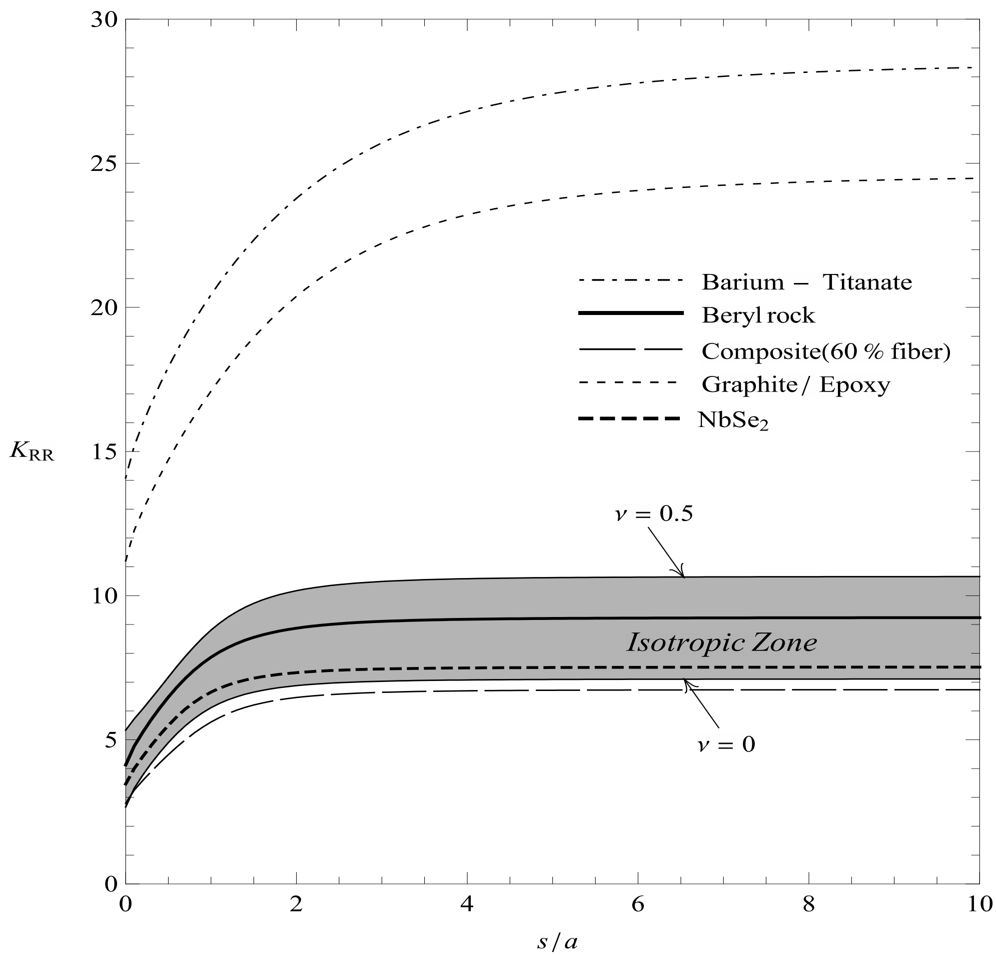


Fig. 2. Rocking stiffness vs. the rigid disk embedment depth for some transversely isotropic materials.

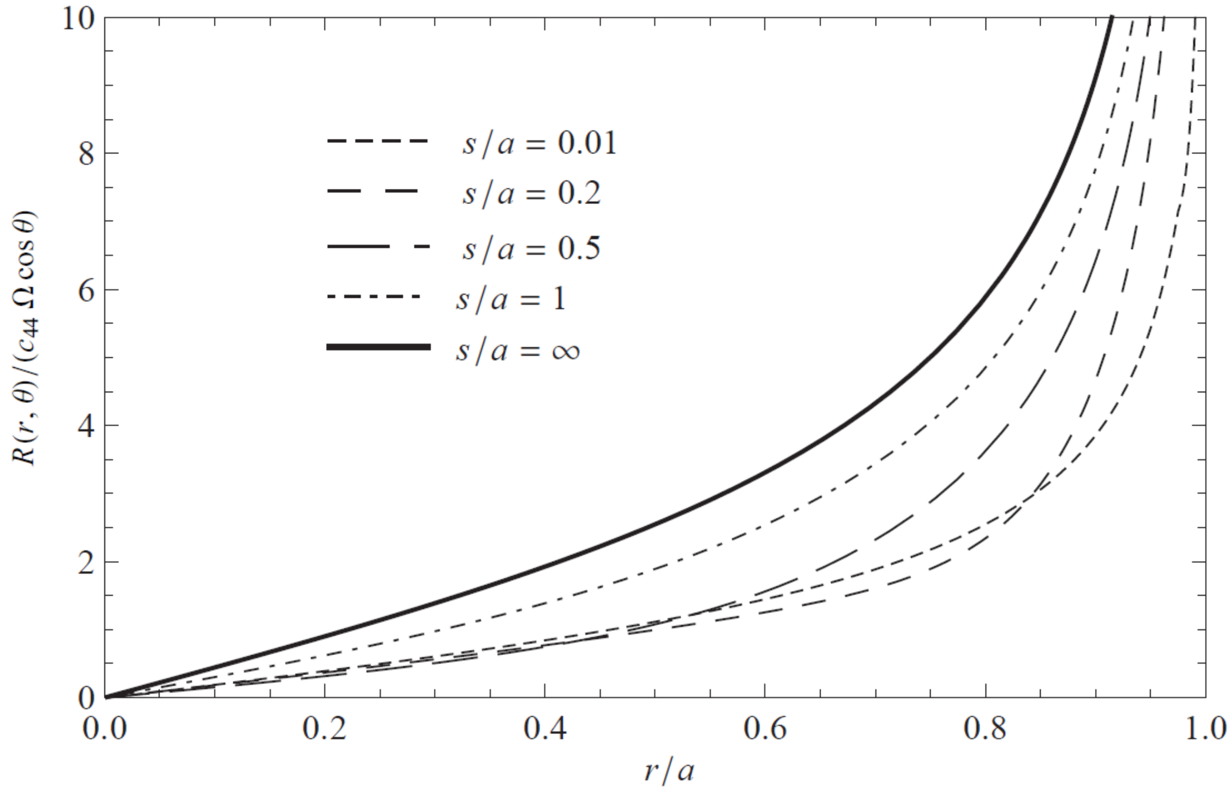
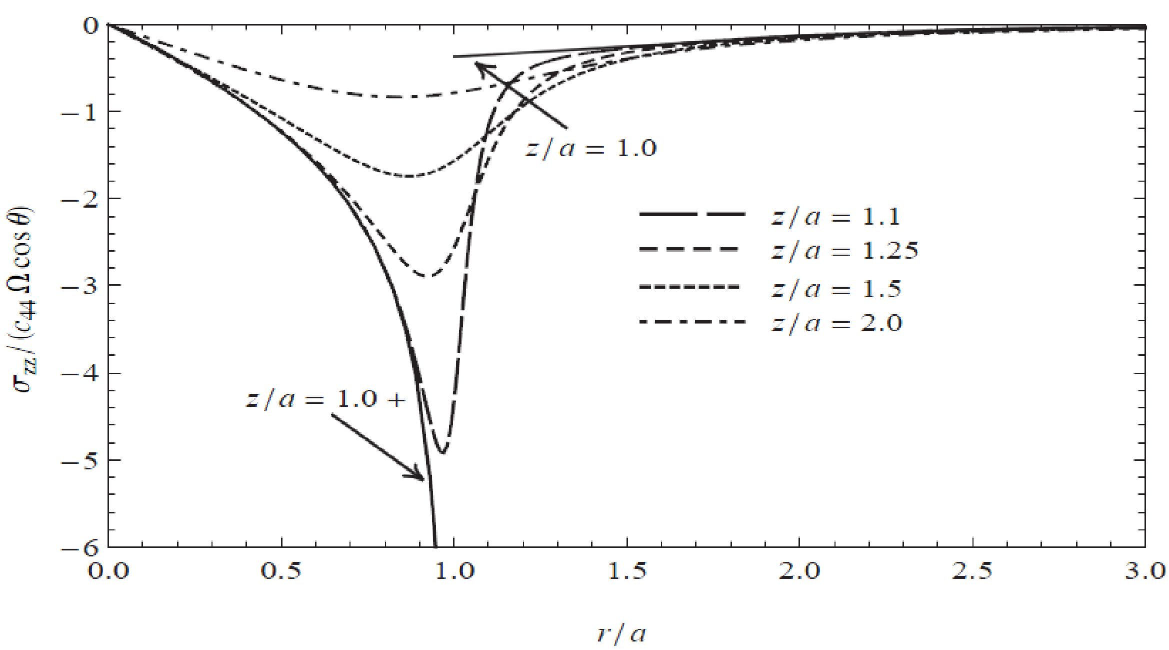


Fig. 3. Contact stress distribution *R(r,θ)* along the radius of a rigid disk embedded in a Beryl rock half-space.

(a)



(b)

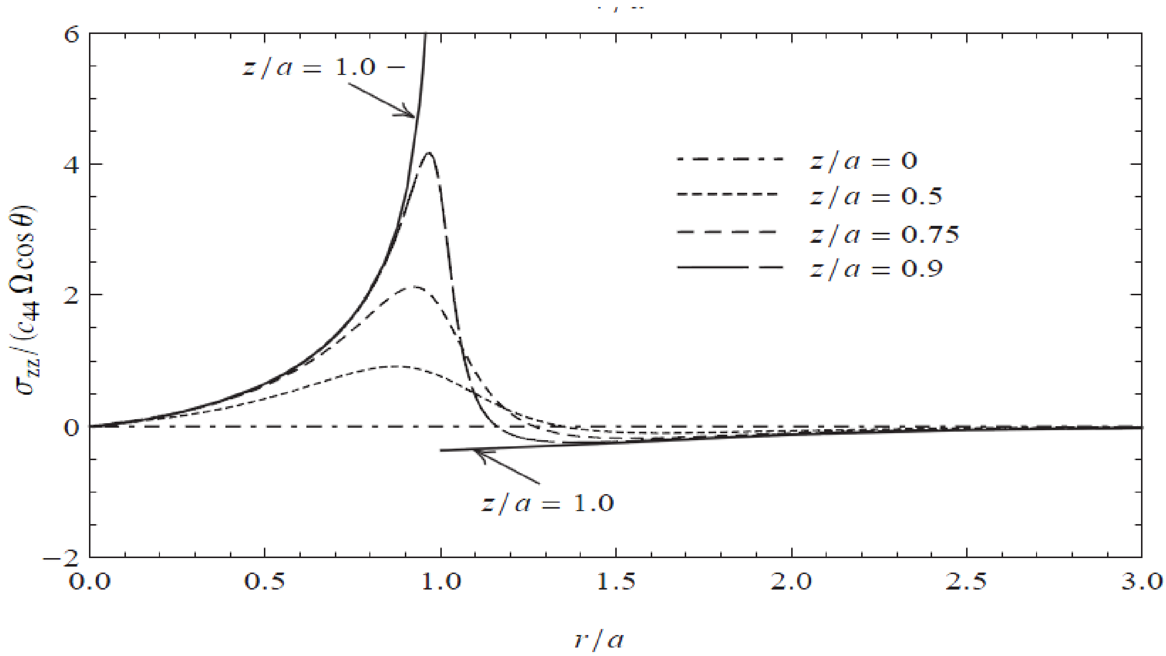
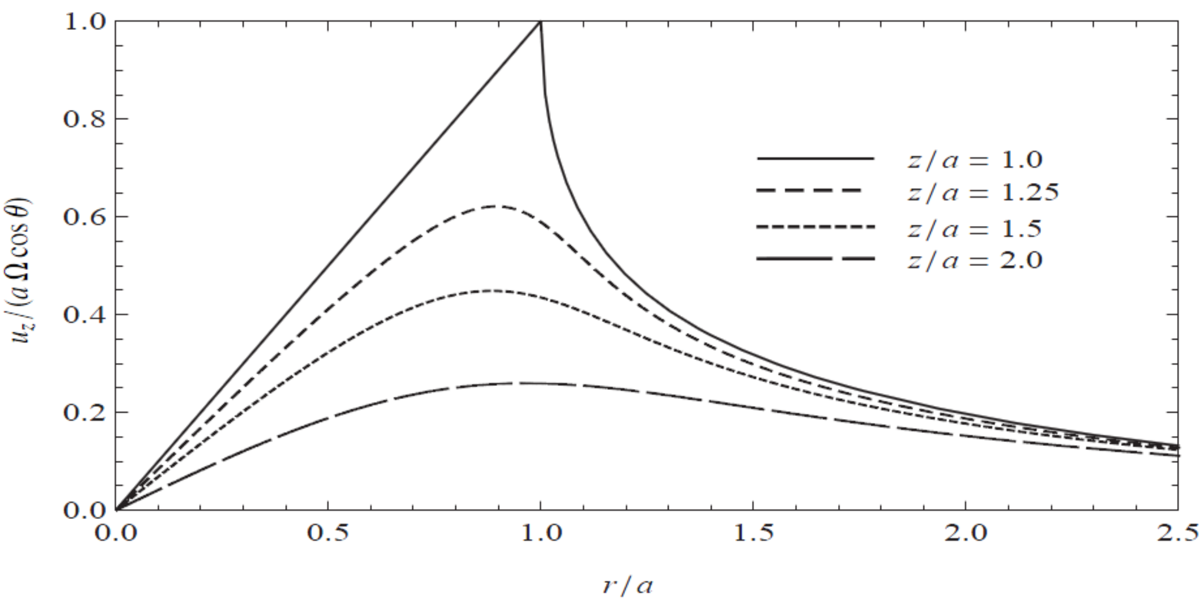


Fig. 4. Normal stress distribution along the radial axis at different depths of a Beryl rock half-space (*s/a=1*).
(a) $z>s$, (b) $z<s$.

(a)



(b)

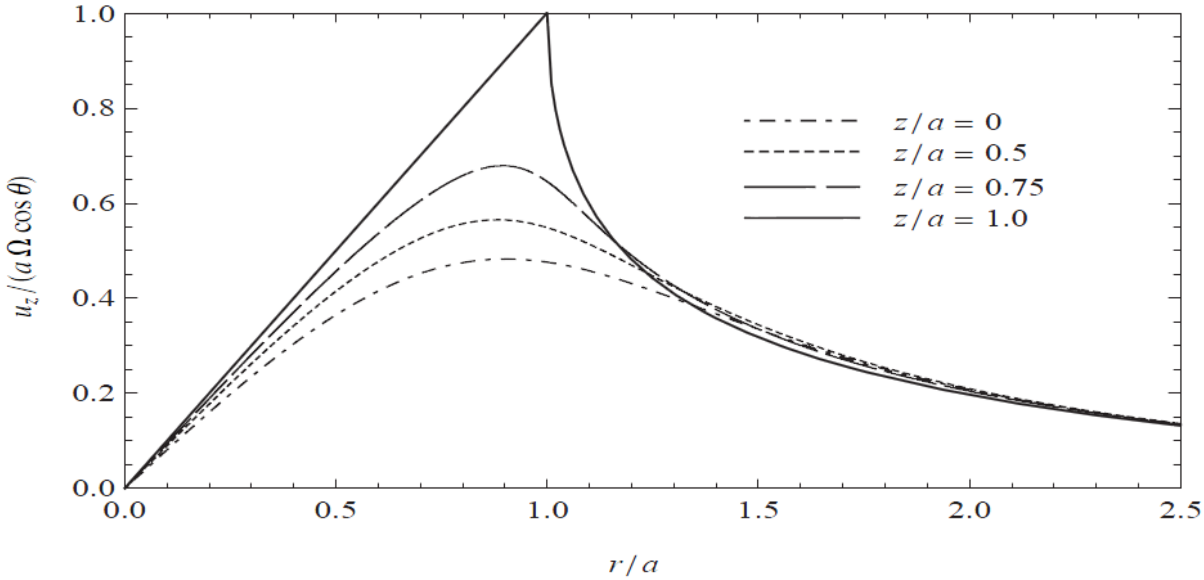


Fig. 5. Normal displacement profile along the radial axis at different depths of a Beryl rock half-space (*s/a=1*). (a) *z≥s*, (b) *z≤s*.

CONCLUSIONS

The asymmetric problem of a rigid disk embedded in a transversely isotropic half-space under the action of a rocking rotation is analytically addressed. By virtue of appropriate Green's functions and Nobel's method, the mixed boundary value problem is reduced to a Fredholm integral equation of the second kind. The contact stress distribution under the disk region and the rocking stiffness is expressed as the solution of the obtained Fredholm integral equation. Some limiting cases pertinent to infinite embedment, surface loading, and isotropic half-spaces are obtained and verified using those available in the literature. The numerical scheme for the solution of the obtained Fredholm integral equation is presented. The rocking stiffness, the normal contact distribution across the disk region, the normal stress distribution, and the vertical displacement are depicted in some dimensionless plots for various transversely isotropic materials.

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