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Analytical Solution for Two-Dimensional Coupled Thermoelastodynamics in a Cylinder

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ABSTRACT: An infinitely long hollow cylinder containing isotropic linear elastic materials is considered to be under the effect of arbitrary boundary stress and thermal condition. The two-dimensional coupled thermoelastodynamic PDEs are specified based on motion and energy equations, which are uncoupled using Deresiewicz-Zorski potential functions. The Laplace integral transform and Bessel-Fourier series are used to derive solutions for the potential functions, then the displacements-, stresses- and temperature-potential relationships are used to determine displacements, stresses and temperature fields. It is shown that the formulation presented here is collapsed on the solution existed in the literature for a simpler case of axis-symmetric configuration. To solve the equation used and evaluate the displacements, stresses and temperature at any point and time, a numerical procedure is needed. In this case, the numerical inversion method proposed by Durbin is applied to evaluate the inverse Laplace transforms of different functions involved in this paper. For numerical inversion, there exist many difficulties such as singular points in the integrand functions, infinite limit of the integral and the time step of integration. Finally, the desired functions are numerically evaluated and the results show that the boundary conditions are accurately satisfied. The numerical evaluations are presented graphically to make engineering sense for the problem involved in this paper for different cases of boundary conditions. The results also indicate that although the thermal induced wave propagates with an infinite velocity, the time lag of receiving stress waves with significant amplitude is not zero. The effect of thermal boundary conditions are shown to be somehow oscillatory, which is due to reflective boundary conditions and may be used in designing of such an element.

Keywords: Bessel-Fourier Series, Coupled Thermoelasticity, Laplace Transform, Numerical Inversion, Potential Functions, Series Expansion, Singular Points.

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INTRODUCTION

The linear theories of thermoelastostatics and thermoelastodynamics have been the subject of numerous investigations for many years. The theory of thermoelasticity is mathematically analogous to the deformation theory of a fluid-saturated porous elastic material, which is often denoted by the theory of consolidation of porous media. Because of this, the solutions for the thermoelastodynamic equations are more general. Considering the coupling behavior in a three-dimensional space, the thermoelastodynamics leads to greater analytical complications than thermoelastostatics. When bodies are subjected to stress or thermal shock and/or loading with high time gradient, the governing equations for the initial boundary value problem are in the form of coupled thermoelastodynamic, which arises from coupling displacement-temperature equations of motion and energy equations (Carlson, 1972). These problems are of great importance in aeronautics, nuclear engineering and military industries.

In the equilibrium theory, where every function is independent of time, body force analogy introduced by Duhamel in 1838 (see Carlson, 1972) has been used to determine the governing equations, which are called displacement-temperature equations of equilibrium, and equilibrium heat equation in elastostatic cases. In this method, a reduced stress and body force are substitutes for the actual stress and body force, respectively (Carlson, 1972). However, this is not a valid theory in thermoelastodynamics. Biot (1956) was the first, who formulated the coupled elastodynamics and energy equations. These are called displacement-temperature equations of motion and coupled heat equation, known as thermoelastodynamic formulations. Nowacki (1959, 1964a, 1964b) proposed a set of potential functions containing a vector function and a scalar function to derive the Green’s function for an infinite thermoelastic medium (see also Nowacki, 1986). Later, Nowacki (1967) proved the completeness of Kaliski-Padstrigacz-Rudiger solution proposed for displacement-temperature equations of motion. Another set of complete solution for the displacement-temperature equations of motion and the coupled heat equation for isotropic materials was given by Deresiewicz and Zorski denoted as the Deresiewicz-Zorski solution (Deresiewicz, 1958; Zorski, 1958; Carlson, 1972).

There exist some mathematical difficulties for solving a fully coupled system of thermoelasticity, containing coupling terms in both equations of motion and energy balance equation. Nickell and Sackman (1968) presented an approximate solution for the coupled thermoelastodynamics with the aid of direct variational method mixed with the general Ritz method. McQuillen and Brull (1970) studied the effect of dynamic parameters (damping and inertia) on the solution of thermoelastodynamics in cylindrical shells using Galerkin method. Li et al. (1983) solved this problem for an axisymmetric cylinder by a finite element method in which spatial and time variables are discretized by the central explicit scheme and a time marching scheme, respectively. Tei-Chen et al. (1989) proposed a finite element model to deal with the transient response in an axisymmetric infinitely long elastic circular cylinder subjected to arbitrary thermal loadings over the cylindrical surface of the domain of interest by the generalized coupled thermoelastic theory. The method proposed by Tei-Chen et al. (1989) was based on the application of the Laplace transform technique, and would utilize the numerical inversion of a transformed solution to obtain a time-domain response.

Eslami and Vahedi (1992) have considered the stress wave in spherical continuum due to a suddenly applied force using a finite element approach. Based on the Radon transform and elements of the distribution theory, Lykotrafitisa and Georgiadis (2003) have developed a procedure to obtain fundamental thermoelastic three-dimensional solutions for thermal and/or mechanical point sources moving steadily over the surface of a half space. In the work of Lykotrafitisa and Georgiadis (2003), the thermal source was defined by a concentrated heat flux, while the mechanical source consisted of normal and tangential concentrated loads.

The current research deals with the equations of coupled thermoelastodynamic in a long hollow cylinder in the case of plane strain with arbitrary traction or thermal boundary conditions applied at inside or/and outside of the cylinder. The thermoelastodynamic equations get uncoupled using the Deresiewicz-Zorski solution (Deresiewicz, 1958; Zorski, 1958), whose completeness (see, for example, Gurtin, 1972) is given in Carlson (1972). The governing equations for potential functions in a polar coordinate system are solved with the aid of Bessel-Fourier series in circumferential direction and Laplace transform with respect to time. Thus, the solutions in the Laplace transform domain are given in the form of Fourier-Bessel series. For validity of the solutions given here, they are degenerated for a simple case of the axisymmetric problem. Because of complexity of the integrands, the Laplace inverse theorem cannot be used analytically, and thus a numerical procedure should be used. After a detailed investigation of different methods for the numerical evaluation of inverse Laplace transform listed in Cohen (2007), the numerical approach proposed by Dubrin (1973) is used in this treatment. To have a deep understanding of the physical phenomena considered here, some illustrations for displacements, stresses and changes of temperature are given in this paper.

MATHEMATICAL MODEL AND THE SOLUTION

Consider a body B in the form of a hollow long cylinder of inner radius and outer radius containing isotropic linear elastic materials to be regular in the sense of Kellogg (1953), (see also Gurtin, 1972) at time t in the time interval , where can be infinite. In the absence of body force and heat source, the classical displacement-temperature equations of motion and the coupled heat equation in the linear isotropic thermoelastic material take the following form (Carlson, 1972):

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) = \gamma\nabla T + \rho\ddot{\mathbf{u}} \tag{1}$$

$$\nabla^2 T - \frac{1}{\kappa}\dot{T} - \eta\,\nabla\cdot\dot{\mathbf{u}} = 0 \tag{2}$$

where $\mathbf{u}$ is the displacement vector, T the change of temperature with respect to the ambient temperature, λ and μ the Lame's constants, ρ the mass density, κ the thermal diffusivity, and ∇ and $\nabla\cdot$ are the operators of gradient and divergence, respectively. In this equation, the superscript dot is used for time derivatives. In Eqs. (1) and (2) the parameters γ and η are defined as:

$$\gamma = (3\lambda+2\mu)\alpha_t = 3k\alpha_t\,, \qquad \eta = \frac{\gamma T_\circ}{\rho c\kappa} = \frac{\gamma T_\circ}{K}\,, \tag{3}$$

where α_t is the thermal expansion coefficient, c the heat capacity, $K = \rho c\kappa$ the thermal conductivity, and k is the bulk modulus. In Eq. (3), $T_\circ$ is a reference temperature. To avoid any difficulty in dealing with the coupled partial differential equations in Eqs. (1) and (2), Deresiewicz-Zorski (Deresiewicz, 1958; Zorski, 1958) proposed a general representation using one scalar and one vector potential function for solving thermoelastodynamic initial-boundary value problems in the isotropic materials. It degenerates to the Helmholtz decomposition in the elastodyamics. Thus, considering the scalar filed ϕ of class $C^{4,3}$ and the vector filed $\boldsymbol{\psi}$ of class $C^{3,2}$ on $\mathcal{B}\times(0,t_0)$, the displacement and temperature fields are given by:

$$\mathbf{u}=\nabla\varphi+\nabla\times\boldsymbol{\psi},\quad T=\frac{\lambda+2\mu}{\gamma}\left(\nabla^2-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)\varphi \qquad (4)$$

where

$$\left(\nabla^2-\frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{\psi}=\mathbf{0} \qquad (5)$$

$$\left[\left(\nabla^2-\frac{1}{\kappa}\frac{\partial}{\partial t}\right)\left(\nabla^2-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)-\eta\frac{\gamma}{\lambda+2\mu}\frac{\partial}{\partial t}\nabla^2\right]\varphi=0 \qquad (6)$$

in which c_1 and c_2 are the longitudinal and shear wave velocities, which are define as:

$$c_1=\sqrt{\frac{\lambda+2\mu}{\rho}},\quad c_2=\sqrt{\frac{\mu}{\rho}} \qquad (7)$$

and $\nabla^2=\partial(r\,\partial/\partial r)/r\,\partial r+\partial^2/r^2\partial\theta^2$ is the two dimensional Laplace operator in any plane perpendicular to the z-axis of the cylindrical coordinate system. As seen in Figure 1, the domain of interest is a long hollow cylinder of inner radius r_1 and outer radius r_2. A cylindrical coordinate system is attached to the domain as shown in Figure 1.

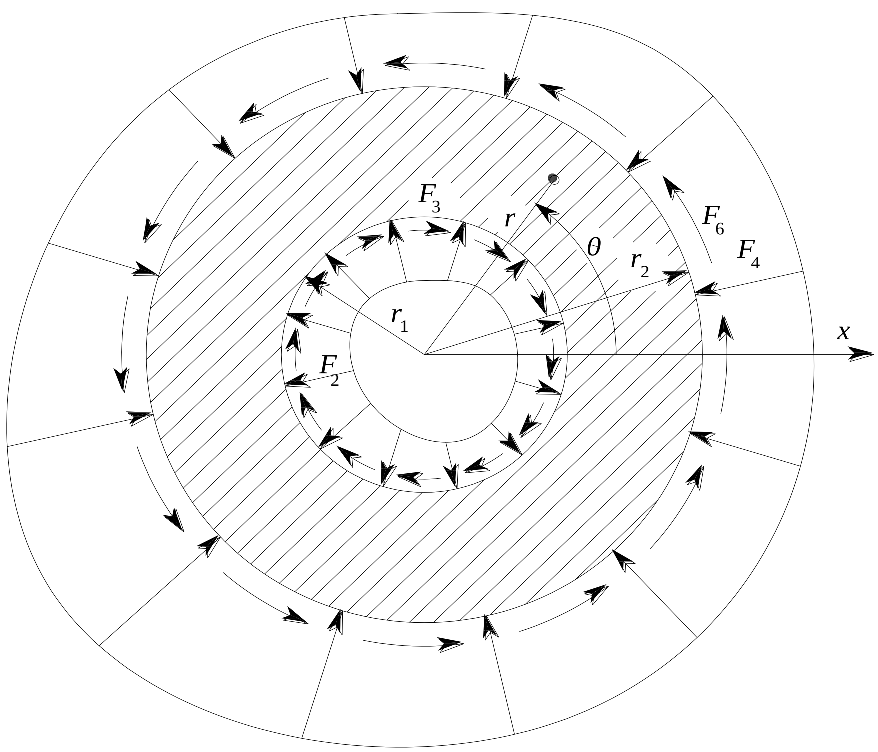

Fig. 1. Configuration of The Problem.

To proceed the solution of partial differential Eqs. (5) and (6), some dimensionless variables should be initially introduced:

$$\hat{r}=\frac{r}{r_1},\ \ \hat{t}=t\frac{c_1}{r_1},\ \ \hat{u}_r(\hat{r},\theta,\hat{t})=\frac{u_r(r,\theta,t)}{r_1},$$
$$\hat{u}_\theta(\hat{r},\theta,\hat{t})=\frac{u_\theta(r,\theta,t)}{r_1},\ \ \hat{T}(\hat{r},\theta,\hat{t})=\frac{T(r,\theta,t)}{T_\circ},\tag{8}$$

where $T_\circ$ is the ambient temperature, and and u_r are u_θ the radial and tangential components of displacement vector. With the use of these dimensionless variables, the Laplace operator and the time derivative are in the form of $\hat{\nabla}^2=\partial/\hat{r}\partial\hat{r}\left(\hat{r}\,\partial/\partial\hat{r}\right)+\partial^2/\hat{r}^2\partial\theta^2=r_1^{\,2}\,\nabla^2$ and $\partial/\partial\hat{t}=r_1\,\partial/c_1\partial t$, respectively. In this treatment, the Fourier series of a general function $\hat{f}(\hat{r},\theta,\hat{t})$ in the circumferential direction:

$$\hat{f}(\hat{r},\theta,\hat{t})=\sum_{m=-\infty}^{+\infty}\hat{f}_m(\hat{r},\hat{t})e^{im\theta},$$
$$\hat{f}_m(\hat{r},\hat{t})=\frac{1}{2\pi}\int_0^{2\pi}\hat{f}(\hat{r},\theta,\hat{t})e^{-im\theta}d\theta,\tag{9}$$

and the Laplace transform in terms of dimensionless time for $\hat{f}_m(\hat{r},\hat{t})$ (Sneddon, 1951):

$$\overline{f}_m(\hat{r};p)=\int_0^\infty\hat{f}_m(\hat{r},\hat{t})e^{-p\hat{t}}d\hat{t}\tag{10}$$

with the inverse Laplace transform:

$$\hat{f}_m(\hat{r},\hat{t})=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\overline{f}_m(\hat{r};p)e^{p\hat{t}}dp,\tag{11}$$

are used, where p is the Laplace transform parameter and $\delta>0$. Since, the boundary value problem is two-dimensional, $\boldsymbol{\psi}(\hat{r},\theta,\hat{t})$ has only the z-component, which is denoted as $\psi_z(\hat{r},\theta,\hat{t})$. By using the Fourier series and applying the Laplace transform with zero initial conditions on Eqs. (5) and (6), Bessel differential equations are generated so that their solutions are derived as:

$$\begin{aligned}\overline{\varphi}_m(\hat{r};p)=&\\ &\sum_{m=-\infty}^{+\infty}\left[A_mI_m(\lambda_1\hat{r})+B_mK_m(\lambda_1\hat{r})\right.\\ &\left.+C_mI_m(\lambda_2\hat{r})+D_mK_m(\lambda_2\hat{r})\right]e^{im\theta}\end{aligned}\tag{12}$$

$$\overline{\psi}_{zm}(\hat{r};p)=\sum_{m=-\infty}^{+\infty}\left[E_mI_m(\lambda_3\hat{r})+F_mK_m(\lambda_3\hat{r})\right]e^{im\theta}\tag{13}$$

where I_m and K_m are the modified m^{th} order Bessel functions of the first and the second kind, respectively. $A_m(p)$ to $F_m(p)$ are the functions to be determined by the boundary conditions. In Eq. (13), and the parameters and in Eq. (12) are the roots of characteristic equation which is:

$$\lambda^4-[(\frac{1}{\kappa}+\xi\eta)pr_1c_1+p^2]\lambda^2+\frac{p^3r_1c_1}{\kappa}=0\tag{14}$$

Substituting the functions $\overline{\varphi}_m(\hat{r};p)$ and $\boldsymbol{\psi}_m(\hat{r};p)=(0,0,\overline{\psi}_{zm}(\hat{r};p))$ in the transformed format of Eq. (4), the displacement components and the temperature are obtained in the Fourier-Laplace space as:

$$\begin{aligned}r_1^2\,\overline{u}_r(\hat{r},\theta;p)&\\ =\sum_{m=-\infty}^{+\infty}\Bigg[&A_m\frac{\lambda_1}{2}\left(I_{m-1}(\lambda_1\hat{r})+I_{m+1}(\lambda_1\hat{r})\right)\\ &-B_m\frac{\lambda_1}{2}\left(K_{m-1}(\lambda_1\hat{r})+K_{m+1}(\lambda_1\hat{r})\right)\\ &+C_m\frac{\lambda_2}{2}\left(I_{m-1}(\lambda_2\hat{r})+I_{m+1}(\lambda_2\hat{r})\right)\\ &-D_m\frac{\lambda_2}{2}\left(K_{m-1}(\lambda_2\hat{r})+K_{m+1}(\lambda_2\hat{r})\right)\\ &-\frac{i\,m}{\hat{r}}\left(E_m\,I_m(\lambda_3\hat{r})+F_mI_{m+1}(\lambda_3\hat{r})\right)\Bigg]e^{i\,m\theta}\end{aligned}\tag{15}$$

$$r_1^2\,\bar{u}_\theta(\hat{r},\theta;p)=\sum_{m=-\infty}^{+\infty}\left\{\frac{-im}{\hat{r}}\Big[A_m I_m(\lambda_1\hat{r})+B_m K_m(\lambda_1\hat{r})+C_m I_m(\lambda_2\hat{r})+D_m K_m(\lambda_2\hat{r})\Big]-\frac{\lambda_3}{2}\Big[E_m I_{m-1}(\lambda_3\hat{r})+E_m I_{m+1}(\lambda_3\hat{r})-F_m K_{m-1}(\lambda_3\hat{r})-F_m K_{m+1}(\lambda_3\hat{r})\Big]\right\}e^{im\theta} \tag{16}$$

$$\begin{aligned}r_1^2\xi T_0\bar{T}(\hat{r},\theta;p)=\sum_{m=-\infty}^{+\infty}e^{im\theta}\\
\times\Bigg\{&A_m\left[\frac{\lambda_1^2}{4}\big(I_{m-2}(\lambda_1\hat{r})+2I_m(\lambda_1\hat{r})+I_{m+2}(\lambda_1\hat{r})\big)+\frac{\lambda_1}{2\hat{r}}\big(I_{m-1}(\lambda_1\hat{r})+I_{m+1}(\lambda_1\hat{r})\big)-\left(\frac{m^2}{\hat{r}^2}+p^2\right)I_m(\lambda_1\hat{r})\right]\\
&+B_m\left[\frac{\lambda_1^2}{4}\big(K_{m-2}(\lambda_1\hat{r})+2K_m(\lambda_1\hat{r})+K_{m+2}(\lambda_1\hat{r})\big)-\frac{\lambda_1}{2\hat{r}}\big(K_{m-1}(\lambda_1\hat{r})+K_{m+1}(\lambda_1\hat{r})\big)-\left(\frac{m^2}{\hat{r}^2}+p^2\right)K_m(\lambda_1\hat{r})\right]\\
&+C_m\left[\frac{\lambda_2^2}{4}\big(I_{m-2}(\lambda_2\hat{r})+2I_m(\lambda_2\hat{r})+I_{m+2}(\lambda_2\hat{r})\big)+\frac{\lambda_2}{2\hat{r}}\big(I_{m-1}(\lambda_2\hat{r})+I_{m+1}(\lambda_2\hat{r})\big)-\left(\frac{m^2}{\hat{r}^2}+p^2\right)I_m(\lambda_2\hat{r})\right]\\
&+D_m\left[\frac{\lambda_2^2}{4}\big(K_{m-2}(\lambda_2\hat{r})+2K_m(\lambda_2\hat{r})+K_{m+2}(\lambda_2\hat{r})\big)-\frac{\lambda_2}{2\hat{r}}\big(K_{m-1}(\lambda_2\hat{r})+K_{m+1}(\lambda_2\hat{r})\big)-\left(\frac{m^2}{\hat{r}^2}+p^2\right)K_m(\lambda_2\hat{r})\right]\Bigg\}\end{aligned} \tag{17}$$

In addition, the stresses may be determined with the use of stress-displacement-temperature relationships $\bar{\sigma}_{ij}=2\mu\bar{\varepsilon}_{ij}+(\lambda\bar{e}-\gamma T_\circ\bar{T})\delta_{ij}$, where $\bar{\varepsilon}_{ij}$ is the strain tensor and δ_{ij} is the Kronecker delta. Therefore, the non-zero stress components are:

$$\begin{aligned}r_1^2\bar{\sigma}_{rr}(\hat{r},\theta;p)=\sum_{m=-\infty}^{+\infty}e^{im\theta}\\
\times\Bigg\{&A_m\left[-\mu\frac{\lambda_1}{\hat{r}}\big(I_{m-1}(\lambda_1\hat{r})+I_{m+1}(\lambda_1\hat{r})\big)+\left(2\mu\frac{m^2}{\hat{r}^2}+(\lambda+2\mu)p^2\right)I_m(\lambda_1\hat{r})\right]\\
&+B_m\left[\mu\frac{\lambda_1}{\hat{r}}\big(K_{m-1}(\lambda_1\hat{r})+K_{m+1}(\lambda_1\hat{r})\big)+\left(2\mu\frac{m^2}{\hat{r}^2}+(\lambda+2\mu)p^2\right)K_m(\lambda_1\hat{r})\right]\\
&+C_m\left[-\mu\frac{\lambda_2}{\hat{r}}\big(I_{m-1}(\lambda_2\hat{r})+I_{m+1}(\lambda_2\hat{r})\big)+\left(2\mu\frac{m^2}{\hat{r}^2}+(\lambda+2\mu)p^2\right)I_m(\lambda_2\hat{r})\right]\\
&+D_m\left[\mu\frac{\lambda_2}{\hat{r}}\big(K_{m-1}(\lambda_2\hat{r})+K_{m+1}(\lambda_2\hat{r})\big)+\left(2\mu\frac{m^2}{\hat{r}^2}+(\lambda+2\mu)p^2\right)K_m(\lambda_2\hat{r})\right]\\
&+E_m\left[im\frac{\mu\lambda_3}{\hat{r}}\left(\frac{2}{\hat{r}}I_m(\lambda_3\hat{r})-I_{m-1}(\lambda_3\hat{r})-I_{m+1}(\lambda_3\hat{r})\right)\right]\\
&+F_m\left[im\frac{\mu\lambda_3}{\hat{r}}\left(\frac{2}{\hat{r}}K_m(\lambda_3\hat{r})+K_{m-1}(\lambda_3\hat{r})+K_{m+1}(\lambda_3\hat{r})\right)\right]\Bigg\}\end{aligned} \tag{18}$$

$$\begin{aligned}r_1^2\bar{\sigma}_{\theta\theta}(\hat{r},\theta;p)=\sum_{m=-\infty}^{+\infty}e^{im\theta}\\
\times\Bigg\{&A_m\Big[-\mu(\lambda_1^2/2)\big(I_{m-2}(\lambda_1\hat{r})+2I_m(\lambda_1\hat{r})+I_{m+2}(\lambda_1\hat{r})\big)+(\lambda+2\mu)p^2I_m(\lambda_1\hat{r})\Big]\\
&+B_m\Big[-\mu(\lambda_1^2/2)\big(K_{m-2}(\lambda_1\hat{r})+2K_m(\lambda_1\hat{r})+K_{m+2}(\lambda_1\hat{r})\big)+(\lambda+2\mu)p^2K_m(\lambda_1\hat{r})\Big]\\
&+C_m\Big[-\mu(\lambda_2^2/2)\big(I_{m-2}(\lambda_2\hat{r})+2I_m(\lambda_2\hat{r})+I_{m+2}(\lambda_2\hat{r})\big)+(\lambda+2\mu)p^2I_m(\lambda_2\hat{r})\Big]\\
&+D_m\Big[-\mu(\lambda_2^2/\hat{r})\big(K_{m-2}(\lambda_2\hat{r})+2K_m(\lambda_2\hat{r})+K_{m+2}(\lambda_2\hat{r})\big)+(\lambda+2\mu)p^2K_m(\lambda_2\hat{r})\Big]\\
&+E_m\left[im\frac{\mu\lambda_3}{\hat{r}}\left(I_{m-1}(\lambda_3\hat{r})+I_{m+1}(\lambda_3\hat{r})-\frac{2}{\hat{r}}I_m(\lambda_3\hat{r})\right)\right]\\
&+F_m\left[-im\frac{\mu\lambda_3}{\hat{r}}\left(K_{m-1}(\lambda_3\hat{r})+K_{m+1}(\lambda_3\hat{r})-\frac{2}{\hat{r}}K_m(\lambda_3\hat{r})\right)\right]\Bigg\}\end{aligned} \tag{19}$$

$$r_1^2\bar{\sigma}_{zz}(\hat{r},\theta;p)=\sum_{m=-\infty}^{+\infty}e^{im\theta}$$
$$\times\left\{A_m\left[-\mu\frac{\lambda_1^2}{2}\left(I_{m-2}(\lambda_1\hat{r})+2I_m(\lambda_1\hat{r})+I_{m+2}(\lambda_1\hat{r})\right)\right.\right.$$
$$-\mu\frac{\lambda_1}{\hat{r}}\left(I_{m-1}(\lambda_1\hat{r})+I_{m+1}(\lambda_1\hat{r})\right)$$
$$\left.+\left(2\mu\frac{m^2}{\hat{r}^2}+(\lambda+2\mu)p^2\right)I_m(\lambda_1\hat{r})\right]$$
$$+B_m\left[-\mu\frac{\lambda_1^2}{2}\left(K_{m-2}(\lambda_1\hat{r})+2K_m(\lambda_1\hat{r})+K_{m+2}(\lambda_1\hat{r})\right)\right.$$
$$+\mu\frac{\lambda_1}{\hat{r}}\left(K_{m-1}(\lambda_1\hat{r})+K_{m+1}(\lambda_1\hat{r})\right)$$
$$\left.+\left(2\mu\frac{m^2}{\hat{r}^2}+(\lambda+2\mu)p^2\right)K_m(\lambda_1\hat{r})\right]$$
$$+C_m\left[-\mu\frac{\lambda_2^2}{2}\left(I_{m-2}(\lambda_2\hat{r})+2I_m(\lambda_2\hat{r})+I_{m+2}(\lambda_2\hat{r})\right)\right.$$
$$-\mu\frac{\lambda_2}{\hat{r}}\left(I_{m-1}(\lambda_2\hat{r})+I_{m+1}(\lambda_2\hat{r})\right)$$
$$\left.+\left(2\mu\frac{m^2}{\hat{r}^2}+(\lambda+2\mu)p^2\right)I_m(\lambda_2\hat{r})\right]$$
$$+D_m\left[-\mu\frac{\lambda_2^2}{2}\left(K_{m-2}(\lambda_2\hat{r})+2K_m(\lambda_2\hat{r})+K_{m+2}(\lambda_2\hat{r})\right)\right.$$
$$+\mu\frac{\lambda_2}{\hat{r}}\left(K_{m-1}(\lambda_2\hat{r})+K_{m+1}(\lambda_2\hat{r})\right)$$
$$\left.\left.+\left(2\mu\frac{m^2}{\hat{r}^2}+(\lambda+2\mu)p^2\right)K_m(\lambda_2\hat{r})\right]\right\} \tag{20}$$

$$\frac{r_1^2}{\mu}\bar{\sigma}_{r\theta}(\hat{r},\theta;p)=\sum_{m=-\infty}^{+\infty}e^{im\theta}$$
$$\times\left\{A_m\left[-im\frac{\lambda_1}{\hat{r}}\left(I_{m-1}(\lambda_1\hat{r})+I_{m+1}(\lambda_1\hat{r})-\frac{2}{\hat{r}}I_m(\lambda_1\hat{r})\right)\right]\right.$$
$$+B_m\left[im\frac{\lambda_1}{\hat{r}}\left(K_{m-1}(\lambda_1\hat{r})+K_{m+1}(\lambda_1\hat{r})+\frac{2}{\hat{r}}K_m(\lambda_1\hat{r})\right)\right]$$
$$+C_m\left[-im\frac{\lambda_2}{\hat{r}}\left(I_{m-1}(\lambda_2\hat{r})+I_{m+1}(\lambda_2\hat{r})-\frac{2}{\hat{r}}I_m(\lambda_2\hat{r})\right)\right]$$
$$+D_m\left[im\frac{\lambda_2}{\hat{r}}\left(K_{m-1}(\lambda_2\hat{r})+K_{m+1}(\lambda_2\hat{r})+\frac{2}{\hat{r}}K_m(\lambda_2\hat{r})\right)\right]$$
$$+E_m\left[-\frac{\lambda_3^2}{4}\left(I_{m-2}(\lambda_3\hat{r})+2I_m(\lambda_3\hat{r})+I_{m+2}(\lambda_3\hat{r})\right)\right.$$
$$\left.+\frac{\lambda_3}{2\hat{r}}\left(I_{m-1}(\lambda_3\hat{r})+I_{m+1}(\lambda_3\hat{r})\right)-\frac{m^2}{\hat{r}^2}I_m(\lambda_3\hat{r})\right]$$
$$+F_m\left[-\frac{\lambda_3^2}{4}\left(K_{m-2}(\lambda_3\hat{r})+2K_m(\lambda_3\hat{r})+K_{m+2}(\lambda_3\hat{r})\right)\right.$$
$$\left.\left.-\frac{\lambda_3}{2\hat{r}}\left(K_{m-1}(\lambda_3\hat{r})+K_{m+1}(\lambda_3\hat{r})\right)-\frac{m^2}{\hat{r}^2}K_m(\lambda_3\hat{r})\right]\right\} \tag{21}$$

The boundary conditions at inner and outer radii are considered to be in the form of traction and temperature conditions, which are more applicable especially in the vessel design. The following conditions are considered at inner radius, $r=r_1$, and outer radius, $r=r_2$, (see Figure 1).

$$\begin{aligned}&\sigma_{rr}(r=r_1,\theta,\hat{t})=-F_1(\theta,\hat{t}),\\&\frac{\partial T(r=r_1,\theta,\hat{t})}{\partial r}-\beta_1 T(r=r_1,\theta,\hat{t})=-\beta_1 F_2(\theta,\hat{t}),\\&\sigma_{r\theta}(r=r_1,\theta,\hat{t})=-F_3(\theta,\hat{t})\end{aligned} \tag{22}$$

$$\begin{aligned}&\sigma_{rr}(r=r_2,\theta,\hat{t})=F_4(\theta,\hat{t}),\\&\frac{\partial T(r=r_2,\theta,\hat{t})}{\partial r}-\beta_2 T(r=r_2,\theta,\hat{t})=-\beta_2 F_5(\theta,\hat{t}),\\&\sigma_{r\theta}(r=r_2,\theta,\hat{t})=F_6(\theta,\hat{t})\end{aligned} \tag{23}$$

where $\beta_1=h_1/K$, $\beta_2=h_2/K$, and h_1 and h_2 are the heat transfer coefficients at inner and outer radii, respectively. Eqs. (22) and (23) are thermal boundary conditions, and the functions F_1, F_3, F_4, and F_6 are tractions boundary conditions. The traction conditions are enough to be piecewise continuous functions of time on a finite interval in the range of $\hat{t}>0$.

Writing the boundary conditions (22) and (23) in the form of Fourier series in terms of circumferential direction and taking their Laplace transform, the boundary conditions will be obtained in a transformed space, where one may use the relations (17) to (21) to have a set of six algebraic equations for six unknown functions $A_m(p)$ to $F_m(p)$. By solving these equations and using the results in the relations (15) to (21), one may obtain the displacements, stresses and temperature in the Laplace domain. By virtue of the theorem of inverse Laplace transform, one may write the displacements, stresses and temperature in the form of line integrals as indicated in Eq. (11).

In cases where the boundary conditions are independent from θ, resulting in an axisymmetric situation, the unknown functions $A_m(p)$ to $F_m(p)$ for all $m \neq 0$ are identically zero. This means that the unknown functions will be only the six functions $A_0(p)$ to $F_0(p)$, and thus the non-vanishing displacement and stresses are:

$$r_1^2\,\bar{u}_r(\hat{r};p) = \lambda_1\left[A_0 I_1(\lambda_1\hat{r}) - B_0 K_1(\lambda_1\hat{r})\right] + \lambda_2\left[C_0 I_1(\lambda_2\hat{r}) - D_0 K_1(\lambda_2\hat{r})\right], \tag{24}$$

$$r_1^2 \xi T_0 \bar{T}(\hat{r};p) = (\lambda_1^2 - p^2)\left[A_0\, I_0(\lambda_1\hat{r}) + B_0\, K_0(\lambda_1\hat{r})\right] + (\lambda_2^2 - p^2)\left[C_0\, I_0(\lambda_2\hat{r}) + D_0\, K_0(\lambda_2\hat{r})\right] \tag{25}$$

$$\begin{aligned} r_1^2\bar{\sigma}_{rr}(\hat{r};p) = {} & A_0\left[-2\mu\frac{\lambda_1}{\hat{r}} I_1(\lambda_1\hat{r}) + (\lambda+2\mu)p^2 I_0(\lambda_1\hat{r})\right] \\ & + B_0\left[2\mu\frac{\lambda_1}{\hat{r}} K_1(\lambda_1\hat{r}) + (\lambda+2\mu)p^2 K_0(\lambda_1\hat{r})\right] \\ & + C_0\left[-2\mu\frac{\lambda_2}{\hat{r}} I_1(\lambda_2\hat{r}) + (\lambda+2\mu)p^2 I_0(\lambda_2\hat{r})\right] \\ & + D_0\left[\mu\frac{2\lambda_2}{\hat{r}} K_1(\lambda_2\hat{r}) + (\lambda+2\mu)p^2 K_0(\lambda_2\hat{r})\right], \end{aligned} \tag{26}$$

$$\begin{aligned} r_1^2\bar{\sigma}_{\theta\theta}(\hat{r};p) = {} & A_0\left[\left(p^2(\lambda+2\mu) - 2\mu\lambda_1^2\right) I_0(\lambda_1\hat{r}) + \frac{2\mu\lambda_1}{\hat{r}} I_1(\lambda_1\hat{r})\right] \\ & + B_0\left[\left(p^2(\lambda+2\mu) - 2\mu\lambda_1^2\right) K_0(\lambda_1\hat{r}) - \frac{2\mu\lambda_1}{\hat{r}} K_1(\lambda_1\hat{r})\right] \\ & + C_0\left[\left(p^2(\lambda+2\mu) - 2\mu\lambda_2^2\right) I_0(\lambda_2\hat{r}) + \frac{2\mu\lambda_2}{\hat{r}} I_1(\lambda_2\hat{r})\right] \\ & + D_0\left[\left(p^2(\lambda+2\mu) - 2\mu\lambda_2^2\right) K_0(\lambda_2\hat{r}) - \frac{2\mu\lambda_2}{\hat{r}} K_1(\lambda_2\hat{r})\right], \end{aligned} \tag{27}$$

$$\begin{aligned} r_1^2\bar{\sigma}_{zz}(\hat{r},\theta;p) = {} & \left(p^2(\lambda+2\mu) - 2\mu\lambda_1^2\right)\left[A_0 I_0(\lambda_1\hat{r}) + B_0 K_0(\lambda_1\hat{r})\right] \\ & + \left(p^2(\lambda+2\mu) - 2\mu\lambda_2^2\right)\left[C_0 I_0(\lambda_2\hat{r}) + D_0 K_0(\lambda_2\hat{r})\right]. \end{aligned} \tag{28}$$

where, of course, the displacement u_θ and the stress $\sigma_{r\theta}$ are identically zero. In such cases, all functions are independent from θ, and the boundary conditions (22) and (23) change to the following form:

$$\begin{aligned} & \sigma_{rr}(r=r_1,\hat{t}) = -F_1(\hat{t}), \\ & \frac{\partial T(r=r_1,\hat{t})}{\partial r} - \beta_1 T(r=r_1,\hat{t}) = -\beta_1 F_2(\hat{t}), \end{aligned} \tag{29}$$

$$\begin{aligned} & \sigma_{rr}(r=r_2,\hat{t}) = F_4(\hat{t}), \\ & \frac{\partial T(r=r_2,\hat{t})}{\partial r} - \beta_2 T(r=r_2,\hat{t}) = -\beta_2 F_5(\hat{t}). \end{aligned} \tag{30}$$

The formulations (24) to (28) are exactly the same as the solutions given by Rahimian et al. (1999a, 1999b) and Li et al. (1983).

NUMERICAL EVALUATIONS

As indicated in Eqs. (9) and (10), the Laplace transform of the function $f(t)$ is given as $F(p) = \int_0^{+\infty} e^{-pt} f(t)\,dt$, where p is generally a complex number and the inverse

Laplace transform of $F(p)$ is $f(t)=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} e^{pt}F(p)\,dp$, where the constant $\delta>0$ is the real part of *p*. There exist several methods for numerically inverting the Laplace transforms (Cohen, 2007). In this study, the numerical method proposed by Dubrin (1973) is used. To do so, one may write the complex function $F(p)$ as.

$$F(p)=\int_{\circ}^{+\infty} e^{-\delta t} f(t)\cos\omega t\,dt - i\int_{\circ}^{+\infty} e^{-\delta t} f(t)\sin\omega t\,dt$$
$$=\mathrm{Re}\{F(p)\}+i\,\mathrm{Im}\{F(p)\} \quad (31)$$

where ω is the imaginary part of p i.e. $p=\delta+i\omega$, and δ is a real constant. The function $F(p)$ may be written in the form of $F(\delta+i\omega)=F(\delta,\omega)$. Since $\delta>0$ is constant, the function $F(\delta,\omega)$ may be thought as a function of ω only. Using of the Eq. (31), one may recognize that the real part of $F(\delta,\omega)$ is an even function, while its imaginary part is an odd function. Since δ is constant, one may write $dp=i\,d\omega$, then considering these properties of the real and imaginary parts of $F(p)$, its inverse Laplace transform is written as:

$$f(t)=\frac{e^{\delta t}}{\pi}$$
$$\times\int_{0}^{+\infty}\left(Re\{F(\delta+i\omega)\}\cos\omega t - i\,Im\{F(\delta+i\omega)\}\sin\omega t\right)d\omega \quad (32)$$

Utilizing this relation with the aid of trapezoidal rule, one may use the following series to evaluate the integrations (32) for the function $f(t)$ (Dubrin, 1973).

$$f(t)=\frac{e^{\delta t}}{T}\mathrm{Re}\left[\frac{1}{2}F(\delta)+\sum_{k=1}^{N}F(\delta+ik\frac{\pi}{T})e^{\frac{ik\pi t}{T}}+\varepsilon(N)\right],$$
$$\lim_{N\to\infty}\varepsilon(N)=0 \quad (33)$$

where π/T is the integration step for numerical evaluation, and $e^{\delta t}\varepsilon(N)/T$ is the error. In this relation, δ should be larger than all the real parts of the poles of $F(p)$. By investigating the function $e^{\delta t}\varepsilon(N)/T$, it is noticed that the smaller the value of δ the smaller the error is.

To show the validity and accuracy of the numerical approach selected in this study, some examples are considered here. To do so, a hollow cylinder of inner radius $r_1=.1\,(m)$ and outer radius $r_2=.2\,(m)$, made of isotropic materials with $\lambda=1.15\times10^{11}(N/m^2)$ and $\mu=7.7\times10^{10}(N/m^2)$ and the following properties are considered: $\rho=7880\,(\frac{kg}{m^3})$, $c=502\,(\frac{j}{kg\,K})$, $K=50.2\,(\frac{j}{ms\,K})$, $\alpha_t=11.7\times10^{-6}(\frac{1}{K})$. Moreover, the reference temperature is considered as $T_{\circ}=300\,(K)$. By the properties given here, the coupling term (see Nowacki, 1986) is obtained as $\frac{(3\lambda+2\mu)^2\alpha_t^2 T_{\circ}}{(\lambda+2\mu)\rho c}=0.00961$, which is a large value in practice (Eslami and Vahedi, 1992).

As the first example, the cylinder is considered to be undergone an impact traction at inner radius defined as $F_1=0.003(1-e^{-10\hat{t}})$. The axis-symmetric traction applied in this example is approached to its maximum value approximately at time $\hat{t}=0.6$. This function was considered by Li, et al. (1983).

Figure 2 shows the variation of the radial stress, σ_{rr}, tangential stress, $\sigma_{\theta\theta}$ and the axial stress, σ_{zz} at $r = r_1$ versus non-dimensional time up to $\hat{t} = 10$. In addition, the function $F_1(\theta, \hat{t})$ is plotted in Figure 2 to check the boundary condition at $r = r_1$. As illustrated in Figure 2, the stress boundary condition is satisfied in the best way. As seen, the tangential and axial stresses approach to a maximum value at a dimensionless time of 5. Similarly, Figures 3 and 4 illustrate the stresses at the middle of the thickness of the cylinder and at the outside of the cylinder, respectively. As seen, there exists a very good agreement for the stress boundary condition at $r = r_2$. Since the non-dimensional time is scaled with the travel time of the longitudinal elastic wave through a distance equal to the inner radius, and the non-dimensional distance is scaled with the length equal to the inner radius, it would take one unit of time for the longitudinal wave to travel a unit distance in non-dimensional scales (Li, et al., 1983). This phenomena is observed in both Figures 3 and 4. Moreover, as expected, some wavy behavior is seen for different functions at the middle of the cylinder, which is because of the reflection of the stress wave from inside and outside walls of the cylinder. An illustration for temperature variations at different places of the cylinder, $\hat{r} = 1$, $\hat{r} = 1.5$ and $\hat{r} = 2$ is prepared in Figure 5. As expected, because of high time gradient of the stress boundary condition, there exist some changes of temperature. Moreover, the highest change of temperature is related to the middle of the cylinder, where the heat transfer is the least.

In the second example, consider the same cylinder as used in the example one, however, it is under the effect of asymmetric traction boundary conditions as $\sigma_{rr}(r_1, \theta, \hat{t}) = -F_1 = -0.003(1 - e^{-10\hat{t}})$, $\sigma_{r\theta}(r_1, \theta, \hat{t}) = -F_3 = -0.006 \sin 2\hat{t}$ and $\sigma_{r\theta}(r_2, \theta, \hat{t}) = -F_6 = 0.0015 \sin 2\hat{t}$. In this example, the inside pressure shock is similar to the first example, and the magnitude of shear stress applied on the inner face is four times bigger than the one applied on the outer face. This condition is necessary for the cylinder in order to prevent any rigid body rotation. Figure 6 shows the radial stress, hoop stress, axial stress and the shear stress at $r = r_1$ versus the non-dimensional time. As seen, the boundary conditions are satisfied in the best fit. Again, a big amplification is seen at a time far from the start time, which needs a special attention. Figure 7 shows the displacements at the inner side of the cylinder. A wavy shape is observed in tangential displacement, which is due to $\sigma_{r\theta}$. Figure 8 illustrates different stresses in addition to the functions F_1 and F_6.

In the third and the last example, the same cylinder under both the traction and the thermal effect is used. The stress boundary condition $\sigma_{rr}(r_1, \theta, \hat{t}) = -F_1 = -0.015\hat{t}e^{-2\hat{t}} \cos^2 \theta$ and thermal boundary condition as

$$\frac{\partial T(r_1, \theta, \hat{t})}{\partial r} - \beta_1 T(r_1, \theta, \hat{t}) = -0.0001\beta_1 J_1(\hat{t}) \sin \theta$$

are considered in this example. The function $J_1(\hat{t})$ in the thermal boundary condition is the Bessel function of the first kind and the first order. Figure 9 shows the different stresses and the boundary conditions versus $\hat{t}$ at $\hat{r} = 1.5$ and $\theta = 0$, and Figure 10 illustrates the same functions at $\hat{r} = 1$ and $\theta = \pi / 2$. As seen, in Figure 10, the stresses $\sigma_{\theta\theta}$ and σ_{zz} are equal to each other. The travel time of different stresses and the maximum values for these functions are clearly observed in Figure 9. Figure 11 illustrates the variation of temperature at $\theta = \pi / 2$, and at $\hat{r} = 1$ and $\hat{r} = 2$. The change of temperature from inside of the cylinder to outside of it is clear.

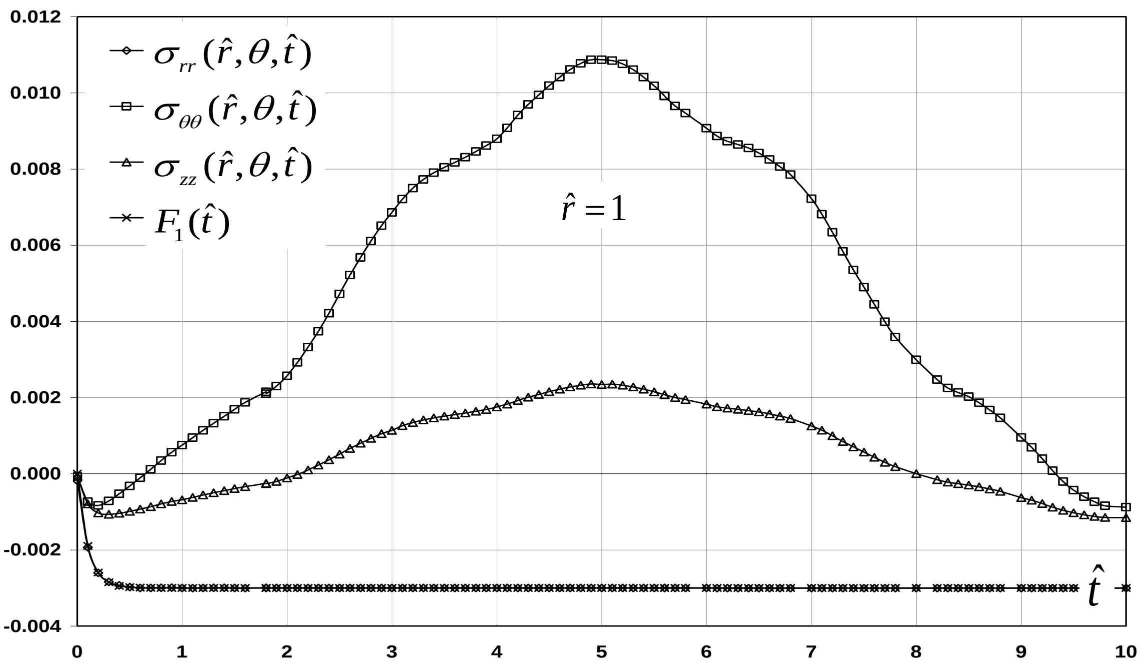

Fig. 2. Time variation of radial, axial and hoop stress accompanied by the stress boundary condition at $\hat{r}=1$ (Example 1).

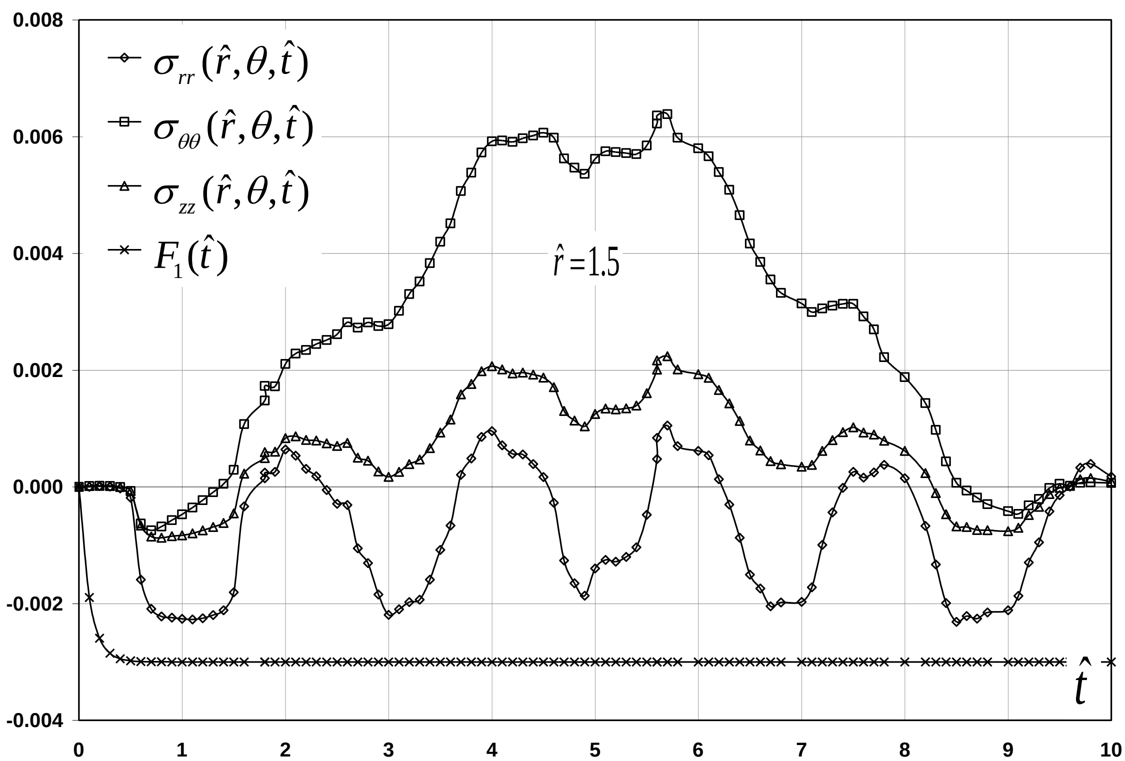

Fig. 3. Time variation of radial, axial and hoop stress accompanied by the stress boundary condition at $\hat{r}=1.5$ (Example 1).

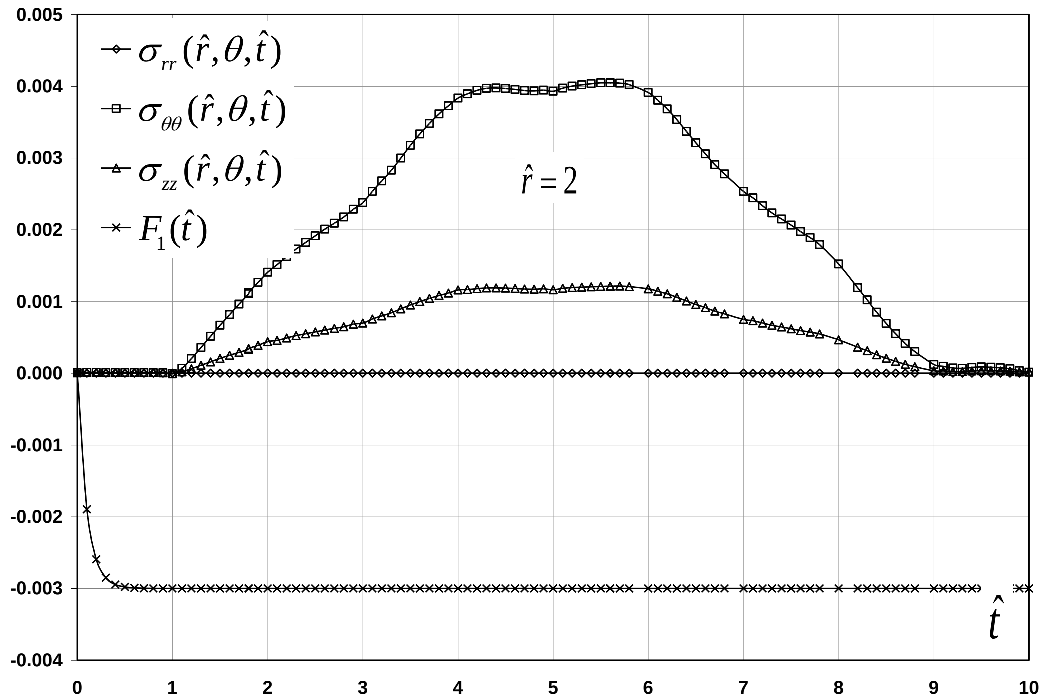

Fig. 4. Time variation of radial, axial and hoop stress accompanied by the stress boundary condition at $\hat{r}=2$ (Example 1).

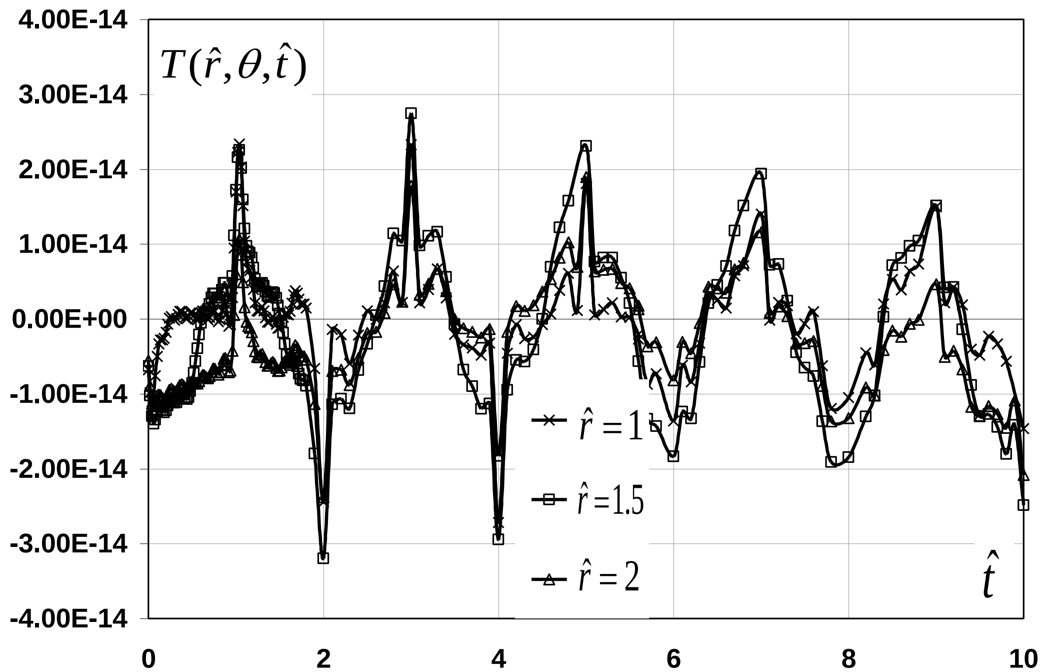

Fig. 5. Time variation of temperature at different place of the cylinder (Example 1).

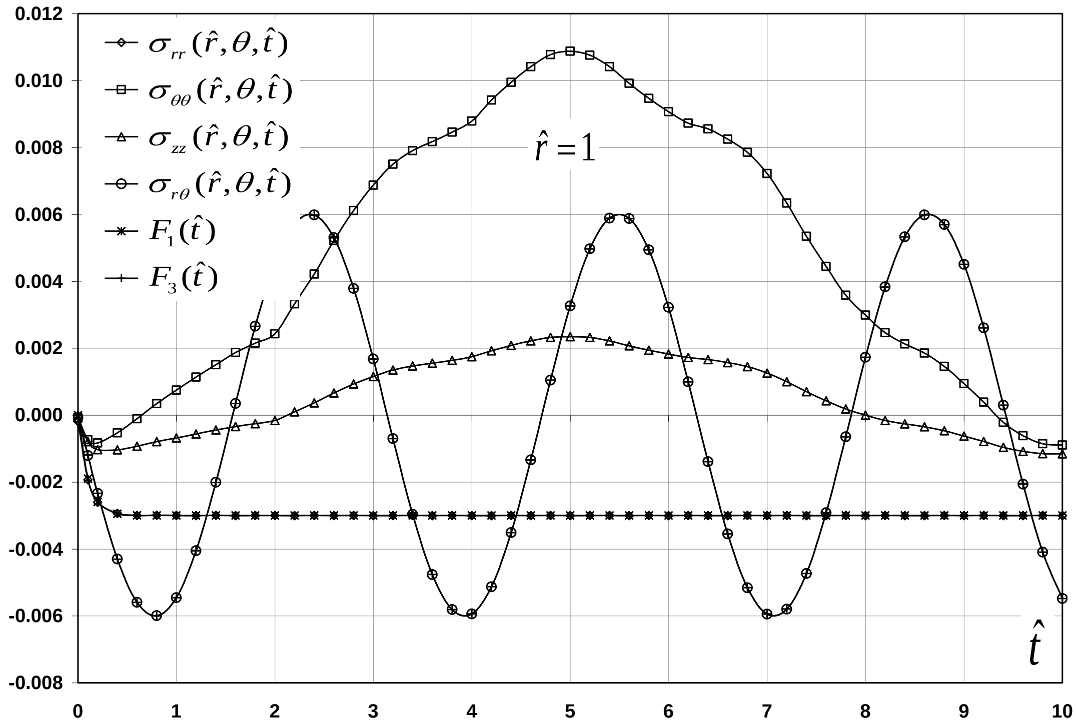

Fig. 6. Time variation of radial, axial and hoop stress accompanied by the stress boundary conditions at $\hat{r}=1$ (Example 2).

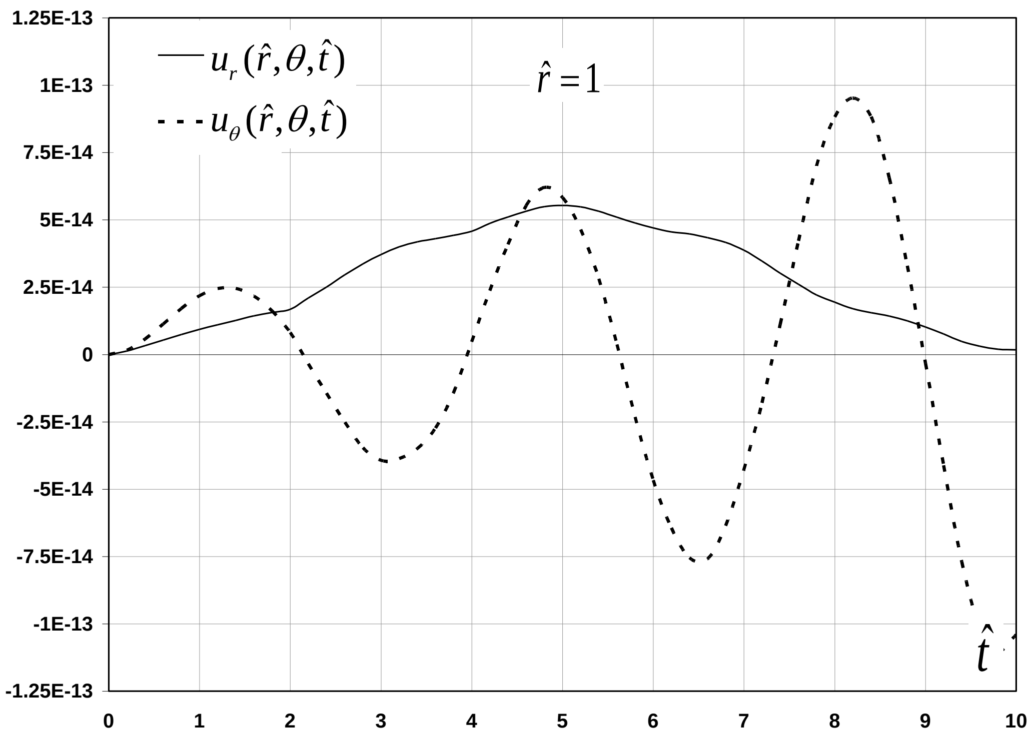

Fig. 7. Time variation of radial and tangential displacements at $\hat{r}=1$ (Example 2).

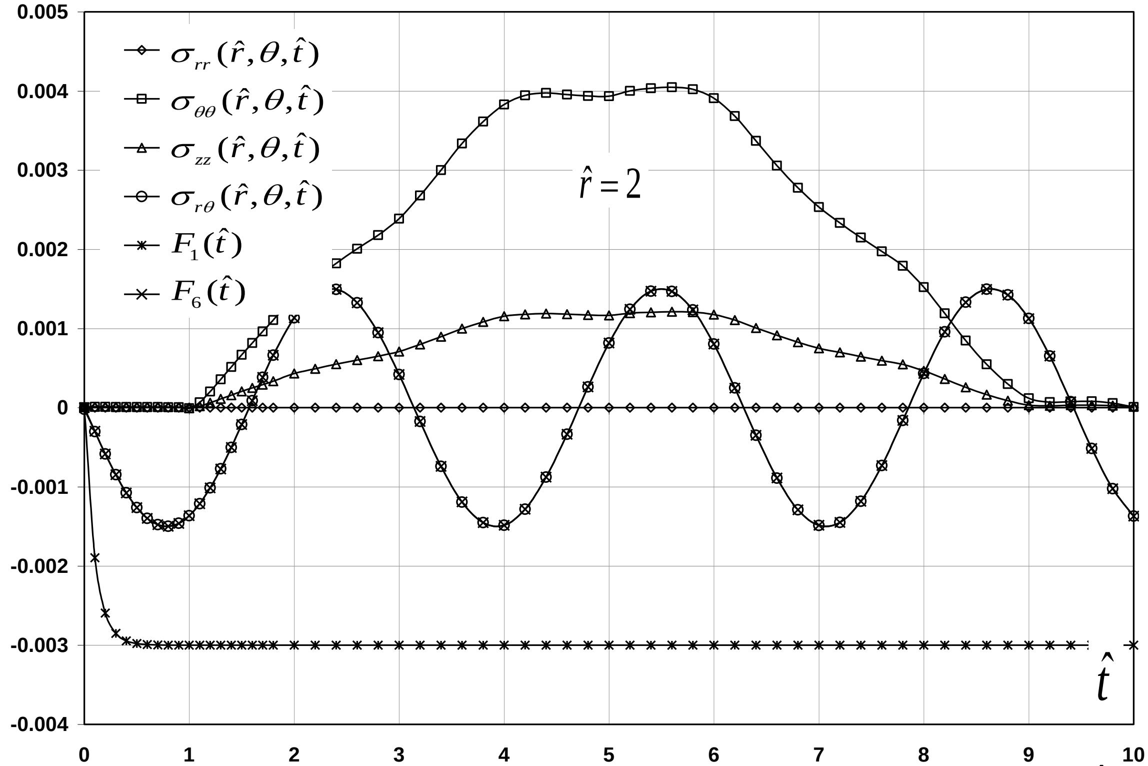

Fig. 8. Time variation of radial, axial and hoop stress accompanied by the stress boundary condition at $\hat{r}=2$ (Example 2).

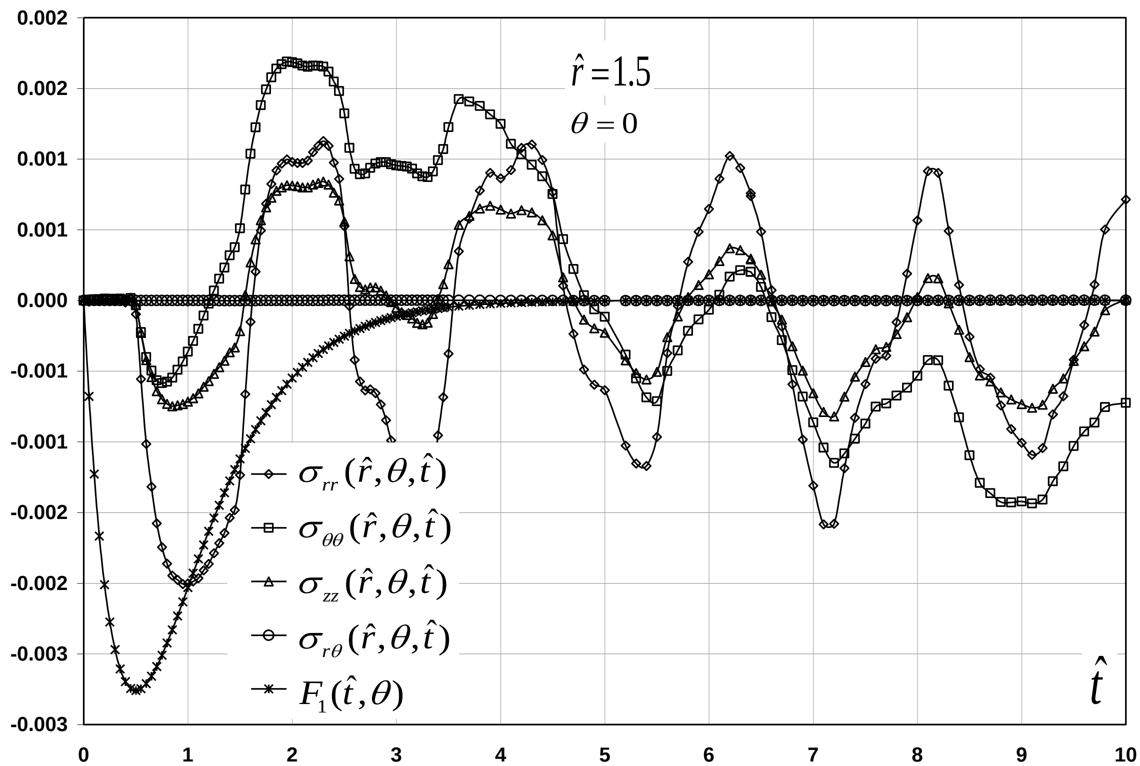

Fig. 9. Time variation of radial, axial and hoop stress accompanied by the stress boundary condition at $\hat{r}=1.5$ and $\theta=0$ (Example 3).

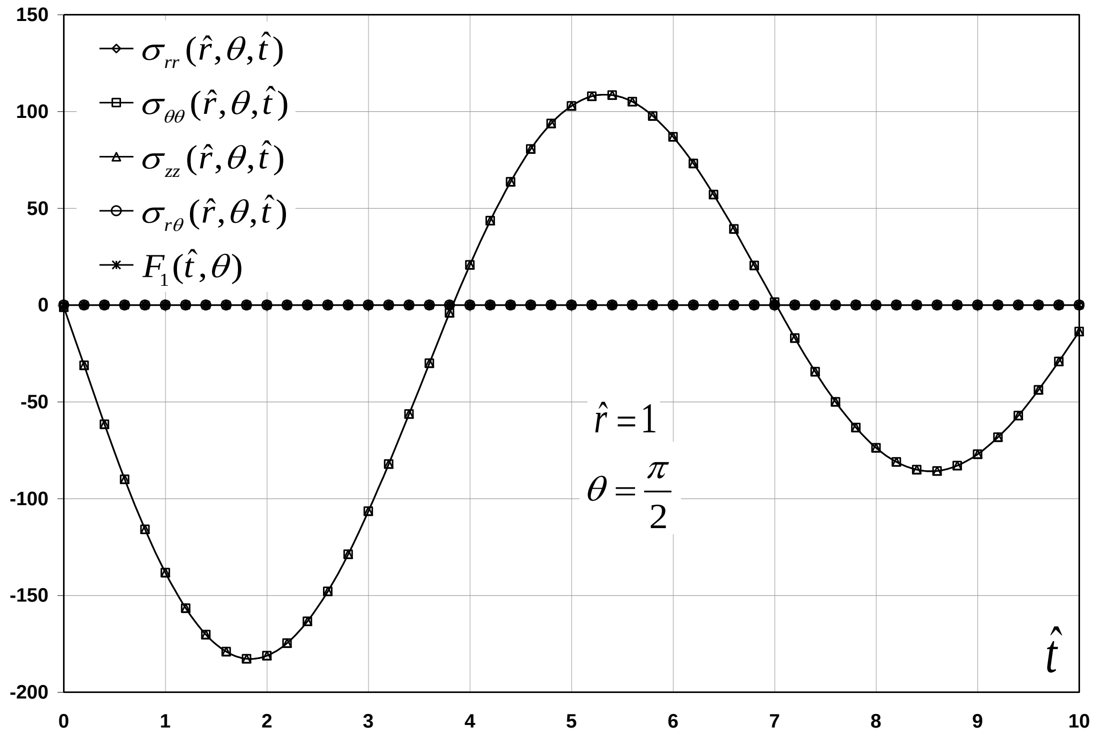

Fig. 10. Time variation of radial, axial and hoop stress accompanied by the stress boundary condition at $\hat{r}=1$ and $\theta=\pi/2$ (Example 3).

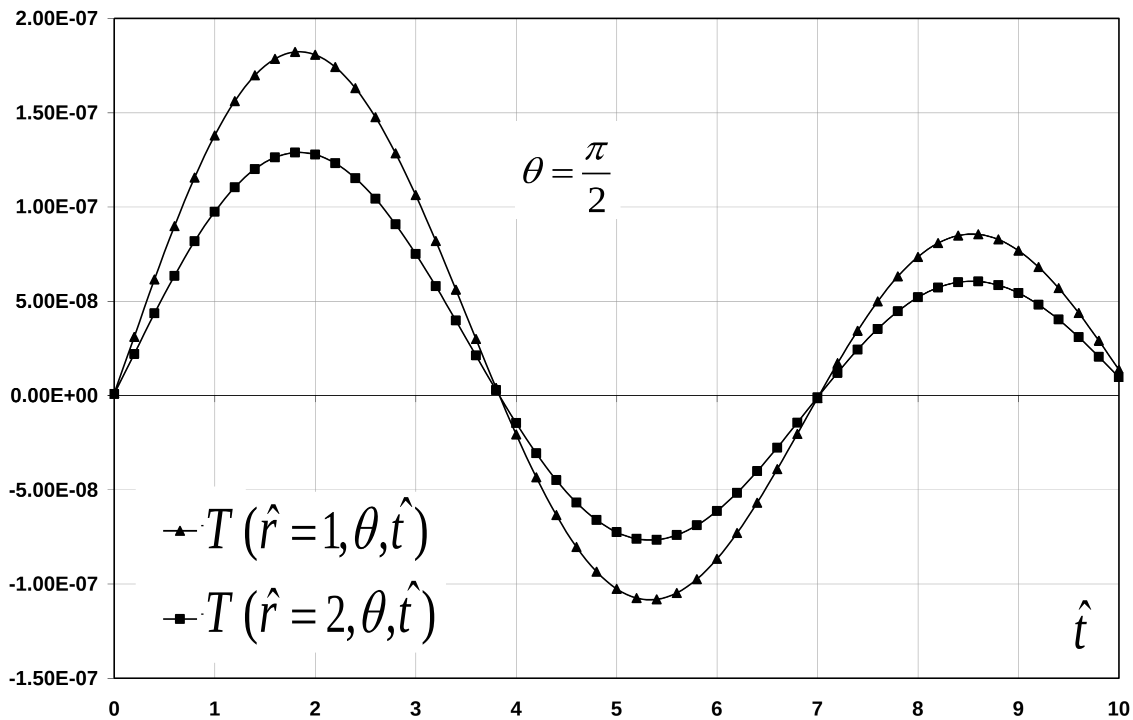

Fig. 11. Time variation of change of temperature at $\theta=\pi/2$ (Example 3).

CONCLUSIONS

A linear thermoelastic solid in the shape of a hollow long cylinder has been considered to be under the effect of an arbitrary boundary stress and arbitrary boundary thermal condition. The two-dimensional coupled thermoelastodynamic partial differential equations have been uncoupled using Nowacki potential functions, and the governing partial differential equations of the potential functions have been solved using the Laplace integral transform and Bessel-Fourier series. With the use of stress-displacement and temperature-potential function relationships in the Laplace-Fourier space, the stresses, displacements and temperature have been analytically determined. By using the Dubrin technique, the numerical inversion of Laplace transform has been carried out to obtain the stresses, displacements and temperature in terms of time. With a very precise attention paid to difficulties existed in the numerical inversion, the desired functions have been numerically evaluated, and associated results show that the boundary conditions have been satisfied very accurately. In general, the results properly revealed the effects of stress boundary conditions on the thermal behavior and vise-versa.

NOMENCLATURE

r_1 = Inner radius
r_2 = Outer radius
c = Heat capacity
h_1 = Heat transfer coefficient of inner surrounding
h_2 = Heat transfer coefficient of outer surrounding
λ = Lame's constant
μ = Lame's constant
T = Change of temperature
ρ = Density
x = Diffusivity
Q = Heat source
$\vec{X}$ = Body force
α_t = Coefficient of thermal expansion
K = Thermal conductivity
k = Bulk modulus
c_2 = Transverse wave velocity
c_1 = Longitudinal wave velocity
$T_\circ$ = Ambient temperature
u_r = Radial displacement
u_θ = Tangential displacement
σ_{rr} = Radial stress
$\sigma_{\theta\theta}$ = Tangential stress
σ_{zz} = Axial stress
$\sigma_{r\theta}$ = Shearing stress
p = Laplace parameter
$\hat{r}$ = Nondimensional radius
$\hat{t}$ = Nondimensional time
$\hat{u}_r$ = Nondimensional radial displacement
$\hat{u}_\theta$ = Nondimensional tangential displacement
δ = Coupling term

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